# THE TETRAHEDRAL (OR 6j) SYMBOL

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Abstract. We will attach a scalar invariant to a tetrahedron whose edges are labelled by irreducible representations of a ternary orthogonal group $SO_3$ over a local field. This generalizes the 6j symbol whose theory was developed by Racah, Wigner, and Regge.

We give several formulas for this invariant, including in terms of hypergeometric-type integrals and functions, and show that it admits a symmetry by the the 23040-element Weyl group of $\mathrm{Spin}_{12}$. We then interpret these results in terms of relative Langlands duality, where the dual story comes from the action of $\mathrm{Spin}_{12}$ on a 16-dimensional cone of spinors.

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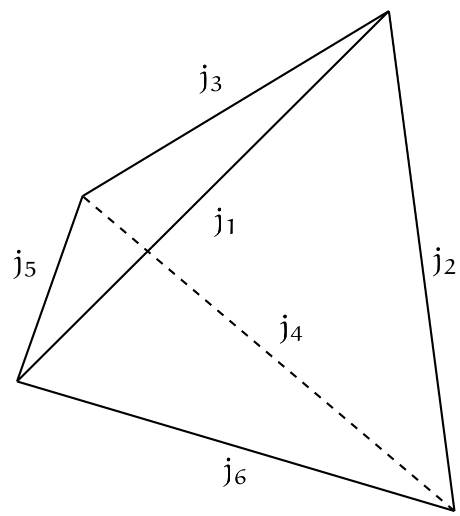



FIGURE 1. A tetrahedron with labeled edges

## 1. INTRODUCTION

The 6j symbol $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$ is an invariant attached to a tetrahedron $\mathcal{T}$ with integral side lengths $j_i$, where the upper and lower rows index opposite edges (as in Figure 1). In truth, each j is indexing the irreducible $(2j+1)$-dimensional representation $V_j$ of the Euclidean rotation group $SO_3$.[1] The origin of the 6j symbol lies in the work of Racah and Wigner on addition theorems for (quantum) angular momentum. The theory was further developed by Regge, who discovered the extraordinary "Regge symmetries" [Reg59] of 6j symbols, and from these deduced new geometric symmetries of tetrahedra [PR69, Rob99]. Since then, the 6j symbol has resurfaced in special function theory, topology and number theory (see § 9 for some references). We will review the classical definition, and then proceed to discuss in more detail what we do in this paper.

### 1.1. The classical definition.

> *. . . I hardly ever take up Dr. Frankland's exceedingly valuable "Notes for Chemical Students," which are drawn up exclusively on the basis of Kekulé's exquisite conception of valence, without deriving suggestions for new researches in the theory of algebraical forms.* — James Joseph Sylvester, *"Chemistry and Algebra."*

These words of Sylvester relate to a graphical calculus for invariant theory. The definition of the 6j symbol is animated by the same spirit; were some strange beast, whose only language was the invariant theory of binary forms, to be confronted with the idea of a tetrahedron, we think it would surely rediscover the definition that follows.

We can realize the irreducible representation $V_j$ of $SO_3$ of dimension $2j+1$ by considering the space of homogeneous polynomials of degree j in three variables $x, y, z$, and restricting

[1] In fact, the 6j symbol is defined for half-integral j, which correspond to representations of the double cover $SU_2$ of $SO_3$. For the purposes of this paper, however, restricting to integral j gives a simpler presentation of the theory. This is inessential; see § 9.4 and § 9.5.3 for further discussion.

them to the sphere $x^2 + y^2 + z^2 = 1$. In what follows, it is more convenient to consider only real polynomials. By integrating over the sphere we get a rotation-invariant inner product on $V_j$.

Now, a classical theorem of invariant theory asserts that the triple product

$$V_a \otimes V_b \otimes V_c$$

admits a nonzero $SO_3$-invariant vector if and only if $a$, $b$ and $c$ satisfy the triangle inequality, i.e. form the sides of a Euclidean triangle. When this happens, the invariant vector is unique up to scaling, and can be explicitly described, with reference to the model just discussed, as the dual of the functional

$$(v_1, v_2, v_3) \longmapsto \int_{S^2} v_1 v_2 v_3.$$

If a triangle with integer side lengths, then, indexes an invariant vector, what can we extract from a tetrahedron $\mathcal{T}$ of integer sides? When $j, j', j''$ correspond to the side lengths of a face of $\mathcal{T}$, we can distinguish an invariant vector inside $V_j \otimes V_{j'} \otimes V_{j''}$, which we normalize to have length 1; by working inside the real form, this distinguishes the invariant vector up to sign. Tensoring these vectors together for all four faces, we arrive at a vector inside

$$V_{j_1} \otimes V_{j_1} \otimes V_{j_2} \otimes V_{j_2} \otimes \dots V_{j_6} \otimes V_{j_6}$$

and contract, using the inner product on each $V_e$. We arrive at a real-valued invariant; this is, up to sign, the classical 6j symbol.

**1.2. What we do in this paper, and why.** Our goal is to set up and study the definitions above in a broader context.

**1.2.1.** *What?* First of all, we will allow $SO_3$ to mean the automorphisms of *any* nondegenerate ternary quadratic form; thus, for example, we allow also $x^2+y^2-z^2$, which results in a noncompact group $SO_{2,1}(\mathbb{R})$.[2] The result of this substitution is that the j-parameters now can vary continuously; informally, the input may be either a Euclidean or a Lorentzian tetrahedron.

Secondly, and perhaps more disorienting to the reader familiar with the classical definition, we will allow the real numbers to be replaced by any local field $F$, for example, the complex numbers or the $p$-adic numbers. Informally speaking, this further enlarges the domain of permissible j-parameters.

We propose to rename the symbol, in this context at least, the "tetrahedral symbol," which seems more evocative than the traditional name, and, at least, does no further injustice to the pioneers.

[2]This group may be more familiar in its isomorphic realization the group $PGL_2(\mathbb{R})$ of projective linear transformations of the plane.

**1.2.2.** *Why?* It seems to us that, in this more general context, the theory both becomes richer in its own right, and also acquires interesting new connections to other areas of mathematics. Thus, for example:

(a) As alluded to above, the 6j symbol possesses an unexpectedly large symmetry group — by a group of order 144, isomorphic to $\mathfrak{S}_4 \times \mathfrak{S}_3$ (where $\mathfrak{S}_n$ is the symmetric group of degree $n$). In the general context, the tetrahedral symbol will acquire symmetry under a much larger group of order 23040, which is isomorphic to the the Weyl group of $\mathrm{Spin}_{12}$.

(b) The tetrahedral symbol in the general context possesses a variety of novel and beautiful integral representations, some of which seem to us much simpler than any integral representation for the original 6j symbol.

(c) When $F$ is a local field, and the representations in question are unramified, it becomes possible to evaluate *explicitly* the tetrahedral symbol in terms of the geometry of a certain remarkable spinor cone on which the group $\mathrm{Spin}_{12}$ acts. In this way, we see the spin group itself, rather than only the Weyl group that was manifested in (a).

(d) The setting in which we develop the theory — namely, the representation of real and $p$-adic groups — is also the setting of the theory of automorphic forms and the Langlands program. As we will see, the tetrahedral symbol has *already* played an interesting unacknowledged role in the former theory — as the kernel underlying certain "spectral reciprocity" formulae; and we will offer various proposals concerning its broader role in the Langlands program.

**1.2.3.** *Connection to existing work.* The idea of generalizing the 6j symbol to the cases of $F = \mathbb{R}$ or $F = \mathbb{C}$ is not a new one. Indeed, the analogues of 6j symbols have been studied for the groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ by several authors, both in the context of special function theory, and of mathematical physics. Among other things, these works define versions of the 6j symbols and give a number of formulas of hypergeometric type, closely related to our § 7 in the case $F = \mathbb{R}$ or $\mathbb{C}$.

We discuss these papers in a little more detail in § 9.4. Broadly speaking, the main point of overlap is point (b) from § 1.2.2. However, our approach to the theory also has a somewhat different emphasis, in that we have sought to give a presentation separating abstract aspects from computational aspects. Thus our definition of the symbol is somewhat different to prior work; it uses no explicit formulas and is manifestly invariant by tetrahedral symmetries. This simplicity comes at a price — more effort is needed to get to explicit formulas.

**1.3. A summary of the paper.** To try to bring out the beauty of the subject matter, we have to some extent separated statements from proofs; in the first part of the paper, the reader will find statements of the theorems, but some proofs are only sketched, with details given in the second part. We summarize briefly the contents of this first part.

- In § 3 we give a more precise version of the discussion above, and explain how to extend it to the case of general $\mathrm{SO}_3$. In this general context, the $V_i$s become infinite-dimensional; nonetheless, there is a simple rearrangement of the definition that avoids analytic difficulty.

- In § 4 we set up various notation that connect tetrahedral geometry to the geometry of the root system $\mathsf{D}_6$ and the associated group $\mathrm{Spin}_{12}$.
- In § 5 we formulate the main theorems:
  - Theorem 5.1.1 proves that the tetrahedral symbol, for principal series, enjoys a $W(\mathsf{D}_6)$-symmetry;
  - Theorem 5.2.1 evaluates the tetrahedral symbol, in the unramified case, in terms of the geometry of a spinor cone.
- In § 6 we give several formulas of geometric nature for the tetrahedral symbol, in particular Proposition 6.1.1 as an integral of characters, Proposition 6.2.1 as an integral of spherical functions, and Proposition 6.3.1 as an integral over moduli of six points on $\mathbb{P}^1$. We regard these formulas as having intrinsic interest, besides their usage to prove the theorems of § 5; the same comment goes for the next section too.
- In § 7 we give hypergeometric formulas for the tetrahedral symbol, in particular Theorem 7.2.1. In the case $\mathsf{F} = \mathbb{R}$ we will express the result as a sum of ${}_4\mathsf{F}_3$ hypergeometric series evaluated at $1$.
- In § 8 we explain how the study of the tetrahedral symbol, and in particular our theorem computing it in terms of a spinor cone, fits into the story of relative Langlands duality.
- In § 9 we rather briefly discuss a number of interesting topics: the unitary integral transform defined by the tetrahedral symbol and its role in number theory; difference equations; and corresponding questions in geometrical representation theory.

**1.4. Acknowledgements.** The first-named author (A.V.) would like to thank Andre Reznikov, for two decades of inspiration and friendship, which included many conversations around the present subject matter; in particular, it was Reznikov's encouragement that led us to really look carefully at the definition of the 6j symbol.

The second-named author (X.G.W.) would like to thank Minh-Tam Trinh for many spontaneous discussions; and his relentless pursuit of creativity in math has always been inspirational throughout the years.

Both of us thank Danii Rudenko for interesting discussions during his visit to IAS. We would also like to thank Tulio Regge for inspiration.

## 2. Review of harmonic analysis on a local field

*The reader should skip this section and refer to it as needed.*

Recall that a *local field* is a field that is equipped with a multiplicative *absolute value* $|\cdot|$, where we require $|\cdot|$ to satisfy the triangle inequality and induce a locally compact topology.[3] Such a field is isomorphic to either the real numbers, the complex numbers, a finite extension of the p-adic numbers, or a finite extension of the field of Laurent series over a finite field.

To facilitate the discussions throughout the paper, it is necessary to introduce some common notations from harmonic analysis of groups defined over a local field. We mainly focus on the case where the group is the multiplicative group $\mathbb{G}_m$.

**2.1. Characters, Haar measures, and absolute values.** Let $F$ be a local field, and if $F$ is nonarchimedean we let $\mathcal{O}$ be its valuation ring with uniformizer $\varpi$ and residue field $k = \mathbb{F}_q$. We fix, once and for all, a nontrivial additive character

$$\Psi = \Psi_F \colon F \longrightarrow \mathbb{C}^\times,$$

as well as an additive Haar measure $d\mu$ on $F$, such that the Fourier transform $\mathcal{F}$ on $F$ with respect to $\Psi$ and $d\mu$ is involutive: in other words, $(\mathcal{F}^2 f)(x) = f(-x)$. In this paper, we will make the following choices for $d\mu$:

(1) when $F$ is nonarchimedean, $\mathcal{O}$ has measure 1;
(2) when $F = \mathbb{R}$, the unit interval $[0, 1]$ has measure 1;
(3) when $F = \mathbb{C}$, the unit square $[0, 1] \times [0, i]$ has measure 1.

We also normalize the absolute value $|\cdot|$ on $F$ to be the factor by which dilation scales the additive Haar measure, or more explicitly:

(1) when $F$ is nonarchimedean, $|\varpi| = q^{-1}$;
(2) when $F = \mathbb{R}$, $|x| = \mathrm{sgn}(x)x$;
(3) when $F = \mathbb{C}$, $|z| = z\bar{z}$.

Then $d\mu/|\cdot|$ is a multiplicative Haar measure on $F^\times$.

Accordingly, the character $\Psi$ is as follows:

(1) when $F$ is $\mathbb{Q}_p$, the choice of uniformizer $\varpi$ induces a group isomorphism $F/\mathcal{O} \cong \mu_{p^\infty}$ (the p-power roots of unity in $\mathbb{C}^\times$), and we let $\Psi(x) = x \bmod \mathcal{O}$;
(2) when $F$ is a finite extension of $\mathbb{Q}_p$, we have $\Psi(x) = \Psi_{\mathbb{Q}_p}(\mathrm{Tr}_{F/\mathbb{Q}_p}(\varpi^{-d}x))$, where the $\mathcal{O}$-module $\varpi^{-d}\mathcal{O}$ is precisely the set of elements $y$ such that $\mathrm{Tr}_{F/\mathbb{Q}_p}(y\mathcal{O}) \subset \mathbb{Z}_p$;
(3) when $F = \mathbb{F}_q(\!(\varpi)\!)$, $\Psi$ is the composition of projecting to the coefficient of $\varpi^{-1}$, and an isomorphism $\mathbb{F}_q \cong \mu_q \subset \mathbb{C}^\times$;
(4) when $F = \mathbb{R}$, $\Psi(x) = e^{-2\pi i x}$;
(5) when $F = \mathbb{C}$, $\Psi(z) = e^{-\pi i(z+\bar{z})} = e^{-2\pi i \Re(z)}$.

Following the conventions of number theory, we shall call a continuous homomorphism $F^\times \to \mathbb{C}^\times$ a *quasi-character* (instead of a character per conventions of group theory) of $F^\times$. A unitary quasi-character (i.e., its image lands in the unit circle) is called a *character* of $F^\times$. The choice of measure on $F^\times$ induces also a measure on the group of characters of $F^\times$ endowed with its natural locally compact topology, in such a way that the Fourier inversion formula holds.

We make the following notational definitions:

[3] As we recall below, the topology in fact determines a canonical absolute value; usually, one therefore thinks of the topology as part of the datum of a local field, but not the absolute value.

**Definition 2.1.1.** Given a quasi-character $\chi\colon F^\times \to \mathbb{C}^\times$ of $F^\times$ and $s \in \mathbb{C}$, we define its *s-twist* to be the quasi-character

$$\chi_s \coloneqq \chi|\cdot|^s.$$

To avoid ambiguity, we adopt the convention that $\chi_s^n \coloneqq (\chi_s)^n = (\chi^n)_{ns}$ for any integer $n$. Namely, *raising to a power always has lower priority than s-twisting.* We also define the shorthands $\chi_+ \coloneqq \chi_{\frac{1}{2}}$, $\chi_- \coloneqq \chi_{-\frac{1}{2}}$, $\chi_+^{-1} \coloneqq (\chi_+)^{-1} = (\chi^{-1})_-$, $\chi_{++} \coloneqq (\chi_+)_+$, and so on.

**2.2. $\gamma$, $L$, and $\epsilon$-factors.** Let $\chi$ be a quasi-character of $F^\times$. There are three meromorphic functions on $\mathbb{C}$ attached to $\chi$: the *$\gamma$-factor*, the *$L$-function* (or *$L$-factor*), and the *$\epsilon$-factor*, which will be used frequently throughout this paper. The most important of the three, for us, will be the $\gamma$-factor.

These will be denoted by $\gamma(s,\chi)$, $L(s,\chi)$ and $\epsilon(s,\chi)$ respectively. They are all compatible with twisting, in the sense that

$$\begin{aligned}
\gamma(s+t,\chi) &= \gamma(s,\chi_t) = \gamma(0,\chi_{s+t}),\\
L(s+t,\chi) &= L(s,\chi_t) = L(0,\chi_{s+t}),\\
\epsilon(s+t,\chi) &= \epsilon(s,\chi_t) = \epsilon(0,\chi_{s+t}),
\end{aligned}$$

and we will denote their values at $s=0$ by $\gamma(\chi)$, $L(\chi)$, $\epsilon(\chi)$, *when defined*, that is to say, when $s=0$ is not a pole of the meromorphic function. When $\chi = |\cdot|^s$, we also use the shorthand $\gamma(s) \coloneqq \gamma(|\cdot|^s)$, and similarly for $L(s)$, and $\epsilon(s)$.

**2.2.1.** *The $\gamma$-factor as the Fourier transform of a multiplicative character.* The most important, for us, will be the $\gamma$-factor $\gamma(s,\chi)$ which tells us what the Fourier transform of a character is. Its value $\gamma(\chi) = \gamma(0,\chi)$ at $s=0$ is characterized by the following equality:

$$\mathcal{F}(\chi^{-1}) = \gamma(\chi)\chi_{-1}. \tag{2.2.1}$$

*A priori*, the left hand side is a distribution; the assertion is that, when $\gamma(s,\chi)$ does not have a pole at $s=0$, the left-hand side is represented by the function on the right-hand side. Homogeneity arguments already imply that $\mathcal{F}(\chi^{-1})$ and $\chi_{-1}$ are multiples of one another, so the only question has to do with the scalar, and that is what $\gamma$ tells us.

*Remark* 2.2.2. The $\gamma$-factor is often characterized in number theory by means of the following equality which is essentially a restatement of (2.2.1):

$$\int_F \check{\Phi}(x)\chi^{-1}(x)|x|^{1-s}\frac{dx}{|x|} = \gamma(s,\chi)\int_F \Phi(x)\chi(x)|x|^s\frac{dx}{|x|},$$

for $\Phi$ a Schwartz-Bruhat function on $F$ (this means a Schwartz function for $F$ archimedean, and a locally constant function of compact support otherwise), and $\check{\Phi} \coloneqq \mathcal{F}(\Phi)$.

**2.2.3.** *Evaluation of the $\gamma$-factor.* It is not difficult to directly evaluate $\gamma(s)$. For example, take the case $F=\mathbb{R}$; one readily computes that, writing $\mathbf{I} = 2\pi i$ and $\bar{\mathbf{I}} = -2\pi i$,

$$\gamma(s) = \frac{1}{(\mathbf{I}^{-s}+\bar{\mathbf{I}}^{-s})\Gamma(s)} = (\mathbf{I}^{s-1}+\bar{\mathbf{I}}^{s-1})\Gamma(1-s). \tag{2.2.2}$$

However, there is a more elegant way to rewrite this as a ratio of two $\Gamma$-funtions that reflects better the involutive property of Fourier transform, and also generalizes well to local fields because $\Gamma$-functions are just special cases of L-functions.

The L-*function* or L-*factor* attached to $\chi$ is defined as:

$$L(s,\chi) := \begin{cases} \frac{1}{1-\chi(\varpi)q^{-s}} & \text{if } F \text{ is nonarchimedean and } \chi \text{ is unramified,} \\ 1 & \text{if } F \text{ is nonarchimedean and } \chi \text{ is ramified,} \\ \pi^{-\frac{s+t+c}{2}}\Gamma\left(\frac{s+t+c}{2}\right) & \text{if } F = \mathbb{R} \text{ and } \chi(x) = |x|^t \mathrm{sgn}(x)^c \text{ where } c \in \{0,1\}, \\ 2(2\pi)^{-(s+t)}\Gamma(s+t) & \text{if } F = \mathbb{C} \text{ and } \chi(z) = |z|^t. \end{cases}$$

Then one always has

$$\gamma(s,\chi) = \frac{L(1-s,\chi^{-1})}{L(s,\chi)} \times (ab^s)$$

for unique $a, b \in \mathbb{C}$; we write $\epsilon(s,\chi) = ab^s$ for this $a$ and $b$.

Said differently, $\epsilon$-factor is defined in terms of the L-factor and the $\gamma$-factor by means of

$$\epsilon(s,\chi) := \gamma(s,\chi)\frac{L(s,\chi)}{L(1-s,\chi^{-1})}. \tag{2.2.3}$$

In the nonarchimedean case, we have, for $\psi$ unramified,

$$\epsilon(s,\chi) = a \cdot q^{-f(s-\frac{1}{2})}$$

where $a$ has absolute value 1. The reader can refer to [Tat79] for more details.

**2.2.4.** *Other equalities for the $\gamma$-factor.* There are a variety of equivalent forms of (2.2.1) that we record for reference. Replacing $\chi$ by $\chi_s$, we have

$$\gamma(s,\chi) = \mathcal{F}(\chi_s^{-1})(1) = \int_F \chi^{-1}(x)|x|^{-s}\Psi(x)dx. \tag{2.2.4}$$

Applying Fourier transform again to (2.2.1) and using the involutive property, we have

$$\check{\chi} := \mathcal{F}(\chi) = \big(\chi(-1)\gamma(1,\chi)\chi_1\big)^{-1}. \tag{2.2.5}$$

which can be written symmetrically as $\mathcal{F}(\chi_-) = \chi(-1)\gamma(\frac{1}{2},\chi)^{-1}\chi_+^{-1}$. If we apply $\mathcal{F}$ again to (2.2.1) we find that

$$\gamma(0,\chi)\gamma(1,\chi^{-1}) = \chi(-1), \tag{2.2.6}$$

and so also $\gamma(\frac{1}{2},\chi)\gamma(\frac{1}{2},\chi^{-1}) = \chi(-1)$, and moreover combining this with (2.2.4) we arrive at

$$\gamma(s)^{-1} = \int_F |x|^{s-1}\Psi(x)dx \tag{2.2.7}$$

**2.2.5.** *Multisets of characters.* Lastly, we use the following notational conventions for *multisets* of characters $X, Y$: we let $X^{-1}$ (resp. $X_s$, $X_+$, $X_-$, etc.) to be the multi-set consisting of characters $\chi^{-1}$ (resp. $\chi_s, \chi_+, \chi_-$, etc.) for $\chi \in X$, and

$$X \otimes Y := \{xy \mid x \in X, y \in Y\}.$$

Similar to the single character case, $X_s^{-1}$ means $(X_s)^{-1}$. If $X = \{\chi\}$ is a singleton, we also use $\chi \otimes Y := X \otimes Y$. In addition, we let

$$\gamma(s, X) := \prod_{\chi \in X} \gamma(s, \chi),$$

and similarly for the L-factor or the $\epsilon$-factor.

For general groups other than $\mathbb{G}_m$, the definition of $\gamma$, $L$, and $\epsilon$-factors are more complicated and we only need them for a small portion of the paper. For this reason we will postpone the discussion until § 9.5.

**2.3. Adjointness and isometric properties of the Fourier transform.** To avoid any sign confusions we write these out. For $\Phi_i$ Schwartz–Bruhat functions on $F$, we have (writing simply $\check{\Phi}$ for the Fourier transform of $\Phi$)

$$\int_F \Phi_1 \check{\Phi}_2 = \int_F \check{\Phi}_1 \Phi_2,$$

as it follows directly from the definition. If we replace above $\Phi_2$ by its Fourier transform, we arrive at

$$\int_F \Phi_1(x)\Phi_2(-x) = \int_F \check{\Phi}_1 \check{\Phi}_2.$$

Finally, replacing $\Phi_2$ by $\overline{\Phi_2(-x)}$, we get

$$\int_F \Phi_1 \overline{\Phi}_2 = \int_F \check{\Phi}_1 \overline{\check{\Phi}_2}.$$

**2.4. Review of integration on projective spaces.** We will several times have occasions to integrate densities over projective spaces, and we now set up relevant notations.

Let $W$ be a $k$-dimensional vector space over $F$. We say a complex-valued function $\varphi$ on $W - \{0\}$ is $(-k)$-homogeneous if $\varphi(tw) = |t|^{-k}\varphi(w)$ for nonzero $t \in F^\times$. There is, up to scaling, a unique $GL(W)$-invariant functional on such functions, denoted by

$$\varphi \longmapsto \int_{\mathbb{P}W} \varphi$$

which we regard as "integration over $\mathbb{P}W$". It can be normalized by the following requirement, once we pick Haar measures on $W$ and $F^\times$: for a Schwartz function $\Phi$ on $W$ itself, the function $\bar{\Phi}(w) = \int_{F^\times} |t|^k \Phi(tx)$ is $(-k)$-homogeneous, and we require

$$\int_{\mathbb{P}W} \bar{\Phi} = \int_W \Phi.$$

Now, fix coordinates $W \simeq F^n$, and suppose the Haar measures on both $W$ and $F^\times$ are induced from a Haar meausre on $F$. Then one readily verifies

$$\int_{\mathbb{P}^{n-1}} \varphi = \int_{F^{n-1}} \varphi(1, x_2, \ldots, x_n) dx_2 \cdots dx_n. \tag{2.4.1}$$

# Part 1. Definitions and statements

## 3. Definition of the tetrahedral symbol

In this section, we define the tetrahedral symbol. The first two subsections, § 3.1 and § 3.2, set up notational preliminaries about tetrahedra and $SO_3$ respectively. In § 3.3 we describe the datum defining the tetrahedral symbol, and the actual definition is given in § 3.4 (under some mild simplifying conditions) and § 3.5 (in general).

**3.1. Tetrahedra.** Consider a tetrahedron, with the following labeling: we label the four vertices by bold face numbers $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$, the six edges by unordered pairs of vertices, and the twelve oriented edges by ordered pairs of distinct vertices; we denote these sets by $\mathbf{V}, \mathbf{E}, \mathbf{O}$ respectively. For two vertices $i, j \in \mathbf{V}$, we will use $ij$ to denote either the associated oriented edge (from $i$ to $j$) or the unoriented one and the context will make it clear which version we are referring to. There are natural maps

$$\mathbf{O} \longrightarrow \mathbf{E}, \quad ij \mapsto ij$$
$$\mathbf{O} \longrightarrow \mathbf{V}, \quad ij \mapsto i$$

which, respectively, assign to an oriented edge the underlying unoriented edge or its source vertex. For each $i \in \mathbf{V}$, we let $\mathbf{O}_i \subset \mathbf{O}$ be the oriented edges with source $i$, that is the preimage of $i$ under the second map above.

**3.2. Group-theoretic setup.** Let

$$\mathcal{R} = SO_3(F)$$

be the special orthogonal group of a nondegenerate ternary quadratic form over the local field $F$. Note that we allow an arbitrary form, not only a split one; therefore, in the case of $F = \mathbb{R}$, the group $\mathcal{R}$ is either compact $SO_3$ or $PGL_2(\mathbb{R})$, and more generally $\mathcal{R}$ is either $PGL_2(F)$ or the projective group of units in a quaternion algebra over $F$. Now define:

$$G = \mathcal{R}^{\mathbf{O}}, \quad D = \mathcal{R}^{\mathbf{E}}, \quad H = \mathcal{R}^{\mathbf{V}}. \tag{3.2.1}$$

Here we regard $D, H$ as subgroups of $G$, by means of the maps $\mathbf{O} \to \mathbf{E}$ and $\mathbf{O} \to \mathbf{V}$. We may visualize elements of $G \simeq \mathcal{R}^{12}$ as 12-tuples of elements of $\mathcal{R}$ thus:

$$\begin{bmatrix} g_{12} & g_{13} & g_{14} & g_{23} & g_{24} & g_{34} \\ g_{21} & g_{31} & g_{41} & g_{32} & g_{42} & g_{43} \end{bmatrix},$$

and then $D \simeq \mathcal{R}^6$ and $H \simeq \mathcal{R}^4$ correspond to subgroups:

$$D = \begin{bmatrix} a & b & c & d & e & f \\ a & b & c & d & e & f \end{bmatrix}, H = \begin{bmatrix} x & x & x & y & y & z \\ y & z & w & z & w & w \end{bmatrix}.$$

We fix a Haar measure on $\mathcal{R}$ and so also on $G, D, H$, etc as follows: if $\mathcal{R}$ is compact, then we let its volume be 1; in the nonarchimedean split case we normalize the Haar measure so that $SO_3(\mathcal{O})$ has volume 1. For the remaining case, when $\mathcal{R} = PGL_2(\mathbb{R})$ or $PGL_2(\mathbb{C})$, we first fix an algebraic volume form on $PGL_2$, identifying it by means of the map $g \mapsto (g \cdot 0, g \cdot 1, g \cdot \infty)$ with $(\mathbb{P}^1)^3$ minus diagonals; on this space, we consider the unique algebraic volume form $\omega$ whose restriction to $(\mathbb{A}^1)^3$ minus diagonals is

$$\omega := \frac{dx_1 \wedge dx_2 \wedge dx_3}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}. \tag{3.2.2}$$

One readily verifies that this this is $\mathcal{R}$-invariant. Then $|\omega|$ defines an volume form on the $F$-points of $\mathcal{R}$. Note that this construction works in the case of $F$ nonarchimedean too, but it gives a *different* measure: it would assign to the $\mathcal{O}$-points the volume $(1 - q^{-2})$.

Sometimes we will need to use both the Haar measure and (3.2.2) for nonarchimedean $F$, and so to emphasize their distinction, we will also use $d^{\mathbb{P}}g$ to denote the measure induced by (3.2.2). For convenience, we define

$$\nu^{\mathbb{P}} := \frac{d^{\mathbb{P}}g}{dg} = \begin{cases} (1 - q^{-2}) & \mathcal{R} = PGL_2(F), F \text{ nonarchimedean}, \\ 1 & \text{otherwise}. \end{cases} \tag{3.2.3}$$

**3.3. The tetrahedral datum.** We denote by $\Pi$ an assignment of *an irreducible smooth representation of $\mathcal{R}$ to each unoriented edge of the tetrahedron.*[4] The representation will, of course, matter only up to isomorphism. To this we will attach an invariant $\{\Pi\}$ which is a complex number defined up to sign. The matter of fixing the sign is an interesting one, which we return to at various points, in particular § 3.6 and part (2) of Theorem 5.1.1.

Denoting the assignment $\Pi$ by $e \mapsto \pi_e$, we will also refer to $\pi_{ij}$ for an oriented edge $ij \in \mathbf{O}$ by means of the natural map $\mathbf{O} \to \mathbf{E}$, and taking the external tensor product of all $\pi_{ij}$ produces an irreducible representation of $G$, denoted simply by $\Pi_G$.

Now, each $\pi_e$ admits an invariant *symmetric* self-pairing

$$(-,-)\colon \pi_e \times \pi_e \longrightarrow \mathbb{C}.$$

Such a pairing always exists ([JL70, Theorems 2.18, 5.11, 6.2]); making clever use of multiplicity one subgroups, Dipendra Prasad proved that it is also always symmetric ([Pra99, Corollary 2, Proposition 2]). Moreover, any two such pairings are equivalent under a rescaling of the underlying space of $\pi_e$; and by Schur's lemma, the automorphisms of the pair $(\pi_e, (-,-))$ reduce to multiplication by $\pm 1$.[5] We will frequently refer to such a pairing as a *rigidification*, because it reduces the automorphism group of $\pi$ from $\mathbb{C}^\times$ to $\pm 1$.

Fix such a self-pairing for each $\pi_e$, which then induces a pairing between $\pi_{ij}$ and $\pi_{ji}$; these induce a $D$-invariant linear *contraction map*

$$\langle - \rangle\colon \Pi_G \longrightarrow \mathbb{C},$$

[4]The word "smooth", in the nonarchimedean case, means that each vector has open stabilizer. In the archimedean case, it connotes that the underlying vector space for $\mathcal{R}$ has a Fréchet topology such that the map $g \mapsto g \cdot v$ is smooth. Subtle issues of topology, however, will be almost irrelevant for us.

[5]While this seems *ad hoc*, this is a special case of a construction that works for any split reductive group, see discussion of duality in [BZSV24].

which, concretely, is given by

$$\bigotimes_{ij\in \mathbf{O}} \nu_{ij} \in \Pi_G \longmapsto \prod_{i<j} (\nu_{ij}, \nu_{ji}).$$

We will also denote this by $\Lambda^D$ to emphasize its D-invariance.

### 3.4. Definition of the tetrahedral symbol when $\Pi_G$ is tempered.

**3.4.1.** *Definition in the compact case.* We begin with the definition of the tetrahedral symbol in "the compact case", that is to say, when $\mathcal{R}$ is compact (cf. § 1.1). In this case all the representations $\pi_{ij}$ are finite dimensional. We put $\pi_{\mathbf{O}_i} := \boxtimes_{ij\in \mathbf{O}_i} \pi_{ij}$, a representation of $\mathcal{R}^{\mathbf{O}_i} \simeq \mathcal{R}^3$. For example,

$$\pi_{\mathbf{O_1}} = \pi_{\mathbf{12}} \boxtimes \pi_{\mathbf{13}} \boxtimes \pi_{\mathbf{14}}.$$

Then $\Pi_G = \boxtimes_{i\in \mathbf{V}} \pi_{\mathbf{O}_i}$, and the pairings on each $\pi_e$ induce also a self-pairing on each $\pi_{\mathbf{O}_i}$. It will be very convenient to make use of the following:

**Lemma 3.4.2.** *There exists a real structure $\pi_e^{\mathbb{R}}$ on which $(-,-)$ defines a real inner product.*

*Proof.* This is well-known, but we write out the argument for later reference. Fix a unitary inner product on $\pi_e$, which we denote by $\langle x, y\rangle$. Necessarily $(x, y) = \langle x, Cy\rangle$ for some complex-antilinear $C : \pi_e \to \pi_e$ which commutes with the group action, and one readily sees that $C^2 = \pm 1$. Symmetry of the pairing implies that $\langle x, Cy\rangle = \langle y, Cx\rangle$; taking $x = Cy$ we deduce that $C^2$ is positive, thus must be $+1$. Then the fixed points of $C$ gives the desired real structure, since $x = \frac{x+Cx}{2} + i\frac{x-Cx}{2i}$. ∎

We suppose that each $\pi_{\mathbf{O}_i}$ admits a nonzero $\mathcal{R}$-invariant vector $\nu_i$; otherwise we will define $\{\Pi\} := 0$.[6] Then the self-pairing $(\nu_i, \nu_i)$ is nonzero, because the self-pairing is positive definite on a real structure for $\pi_{\mathbf{O}_i}$ and a suitable multiple of $\nu_i$ is real for this structure. We therefore normalize $\nu_i$ so that $(\nu_i, \nu_i) = 1$ and let

$$V = \nu_{\mathbf{1}} \otimes \nu_{\mathbf{2}} \otimes \nu_{\mathbf{3}} \otimes \nu_{\mathbf{4}} \in \Pi_G, \tag{3.4.1}$$

which, having fixed pairings, is uniquely specified up to a sign. Contract $V$ to obtain what we shall call the *tetrahedral symbol*

$$\{\Pi\} := \langle V\rangle \in \mathbb{C}. \tag{3.4.2}$$

This depends on our choice of pairings only up to an overall $\pm$ sign.

In the classical case when $\mathcal{R}$ is the compact real $SO_3$ and each $\pi_{ij}$ is indexed by its highest weight, this is up to sign the standard definition of the classical 6j symbol; this will follow from our computations in § 6.1.

[6] It would be more proper to regard the symbol as *undefined* in those cases. We adopt this convention, however, in order to avoid having to repeatedly say "or $\{\Pi\}$ is undefined" in various statements; the same convention is followed in the theory of 6j symbols.

**3.4.3.** *Definition in the tempered case.* We now drop the assumption that $\mathcal{R}$ is compact, but assume that each $\pi_e$ is *tempered*. Recall that an irreducible representation of a group over a local field is called *tempered* if it is "weakly contained" in the regular representation $L^2(\mathcal{R})$; for details on what this means, see [HCH88]. As we discuss further in § 3.5.1, tempered reprsentations can be considered, roughly speaking, as a "real form" of a complex variety parameterizing all representations. In the compact case, all irreducible representations are tempered.

Every tempered representation admits a G-invariant inner product; in particular, by inspecting the proof, we see that Lemma 3.4.2 continues to apply: there exists a real structure $\pi_e^{\mathbb{R}}$ and a $\mathcal{R}$-invariant inner product upon it.

In place of D and H-invariant *vectors* in $\Pi_G$, we will use D and H-invariant *functionals*. The space of D-invariant functionals is always exactly one-dimensional, spanned by $\Lambda^D$ defined as before; the space of H-invariant functionals is zero or one (see [Pra90]). Although unnecessary for our immediate purposes, the two possibilities are distinguished by the signs of various $\epsilon$-factors, as explained in the just-quoted work.

If a nonzero H-invariant functional exists, there are two natural ways to construct it. First of all, we can normalize one, up to sign, by the following rule:

$$\Lambda^H(v_1)\Lambda^H(v_2) = \int_H (hv_1, v_2)dh, \quad v_1, v_2 \in \Pi_G. \tag{3.4.3}$$

*Note the pairing that is used here, and indeed everywhere unless explicitly stated otherwise, is the self-duality pairing, not an inner product.*

The integral is absolutely convergent on account of the assumption that $\Pi_G$ is tempered. In this case, $\Lambda^H$ is nonzero if and only if the space of H-invariant functionals is nonzero, as is proven in a more general context in [SV17]. On the other hand, we can start with $\Lambda^D$ and just *average it to be* H*-invariant*:

$$(\Lambda^H)' \colon v \longmapsto \int_{H \cap D \backslash H} \Lambda^D(hv)dh. \tag{3.4.4}$$

This is also absolutely convergent under the assumption of temperedness: see § 10.3. By the multiplicity-one property, $\Lambda^H$ and $(\Lambda^H)'$ are proportional to one another; we define the tetrahedral symbol $\{\Pi\}$ to be the proportionality factor:

$$(\Lambda^H)' = \{\Pi\}\Lambda^H.$$

This agrees with the definition given in the compact case. Indeed, define V as in (3.4.1); it is straightforward to show that $\Lambda^H = (V, -)$ satisfies (3.4.3). Thus, on the one hand, $\Lambda^H(V) = 1$ by definition, and on the other hand, $(\Lambda^H)'(V) = \langle V \rangle$, which coincides with (3.4.2).

**Warning 3.4.4.** The definition just given is *dual to the definition given in § 1.1*. Relative to that discussion, we have swapped the role of vertices versus faces; or to put it differently, by swapping the upper row with the lower row in the classical 6j notation.

### 3.5. Definition of the tetrahedral symbol in the general case.

**3.5.1.** *Principal series and the classification of tempered representations.* We can extend the definition beyond tempered representations by a process of analytic continuation. This discussion is only relevant in the case when $\mathcal{R}$ is noncompact, which we shall therefore assume.

We must first recall the notion of principal series representation and the classification of tempered representations. Principal series are a twisted version of functions on the projecive line $\mathbb{P}^1_F$. Namely, let $\chi$ be a quasi-character of $F^\times$, and consider the space $\pi_\chi$ of functions on $F^2 - \{0\}$ that are homogeneous of degree $(\chi^2)_{-1} = \chi_-^2 = (\chi_-)^2$, that is, a function on $F^2 - \{0\}$ satisfying

$$f(\lambda \cdot z) = \chi^2(\lambda)|\lambda|^{-1} f(z).$$

Then $\mathcal{R}$, identified with $\mathrm{PGL}_2(F)$, acts on such functions by means of

$$g \cdot f \colon z \longmapsto f(z\tilde{g})\chi_-^{-1}(\det \tilde{g}) = f(z\tilde{g})(\chi^{-1})_{\frac{1}{2}}(\det \tilde{g}),$$

where $\tilde{g} \in \mathrm{GL}_2(F)$ is an arbitrary lift of $g$, the choice of which has no effect on the action. This construction yields an association

$$\text{quasi-characters } \chi \text{ of } F^\times \longrightarrow \text{smooth representations } \pi_\chi \text{ of } \mathcal{R}.$$

The resulting representation $\pi_\chi$ is called a *principal series representation*. It is not irreducible in general, but it is if $\chi$ is away from certain discrete subset of all quasi-characters. Moreover, $\pi_\chi$ and $\pi_{\chi^{-1}}$ have the same semi-simplification. For $\chi$ a character, that is to say, a unitary quasi-character, in particular, $\pi_\chi$ is always irreducible, tempered, and $\pi_\chi \simeq \pi_{\chi^{-1}}$.

With this setup, a tempered representation is either:

(1) of the form $\pi_\chi$, where $\chi$ is a character uniquely determined up to the substitution $\chi \leftrightarrow \chi^{-1}$; or
(2) isomorphic to a direct summand of $L^2(\mathcal{R})$; these form a countable set of irreducible representations called the *discrete series*.

**3.5.2.** *Definition of the tetrahedral symbol in the general case.* We continue to suppose that $\mathcal{R}$ is noncompact. Let $\mathcal{P}_0$ be the set of isomorphism classes of irreducible tempered representations. Because of the classification above, we can think of $\mathcal{P}_0$ as a subset of points of a complex analytic variety $\mathcal{P}$:

$$\mathcal{P} = \{\text{discrete series}\} \coprod \big\{\text{quasi-characters of } F^\times \text{ up to inversion}\big\}. \tag{3.5.1}$$

In fact, $\mathcal{P}_0$ is a subset of the real points $\mathcal{P}(\mathbb{R})$ for a natural real structure on $\mathcal{P}$. What is more important for us is that an analytic function on $\mathcal{P}$ that vanishes on $\mathcal{P}_0$ is identically vanishing; therefore, there is at most one way to extend a function from $\mathcal{P}_0$ to a meromorphic function on $\mathcal{P}$. The tetrahedral symbol thus extends:

**Proposition 3.5.3.** *The function $\{\Pi\}^2$ extends to a meromorphic function on $\mathcal{P}^{\mathbf{E}}$.*

The proof is given in § 10.1. It is based on studying the the asymptotic behavior of the integrand in (3.4.4) that enters into the definition of $\{\Pi\}$, which in all cases is very simple; for example in the nonarchimedean case it is a geometric progression, in suitable coordinates.

*Remark* 3.5.4. Recall that $\{\Pi\}$ is defined only when $\Pi_G$ admits an H-invariant functional. Whether this is so depends only on the component of $\mathcal{P}^{\mathbf{E}}$ to which $\Pi$ belongs. Consequently, the function $\{\Pi\}$ is simply identically zero on some components of $\mathcal{P}^{\mathbf{E}}$; and these can be identified by means of $\epsilon$-factors using the results of Prasad, see § 9.5.1.

*Remark* 3.5.5. One way to interpret the proposition as giving an extension of tetrahedral symbol from tempered representations to all irreducible ones. Indeed, there is an identification of sets

$$\mathcal{P} \simeq \{\text{isomorphism classes of irreducible representations of } \mathcal{R}\} \tag{3.5.2}$$

defined as follows: we associate to $\chi$ as above $\pi_\chi$ if it is irreducible, and, otherwise, the unique finite-dimensional subquotient of $\pi_\chi$ ([JL70, Theorems 3.3, 5.11, 6.2]). It is likely that the function $\{\Pi\}$ thus extended coincides, even for nontempered representations, with a function defined by means of suitably regularizing the integrals appearing in § 3.4.3. However, we do not examine this in the current paper, and the identification (3.5.2) will play no further role.

**3.6. The sign ambiguity.** Note that, even when $\{\Pi\}$ is defined, $\{\Pi\}$ is defined only up to a sign. Unlike the classical situation, *this sign ambiguity is essential*: the meromorphic function $\{\Pi\}^2$ described in the above proposition does not in general admit a meromorphic square root. Of course, we could redefine the symbol to be $\{\Pi\}^2$ instead of $\{\Pi\}$ but we prefer not to do so for two reasons: first of all, it is $\{\Pi\}$ and not its square that corresponds to the 6j symbol; and secondly, the choice of sign is actually very interesting.

As a general convention, when we prove a formula of the form

$$\{\Pi\} = \cdots,$$

we always regard the equality as being up to sign. However, in all important such instances, and particularly in §§ 5–7, *the formula in fact gives more:* it gives a mechanism to resolve the sign ambiguity, in the sense that we will produce an explicit meromorphic function $\gamma$ which belongs to the same square class as $\{\Pi\}^2$, and then $\sqrt{\gamma}\cdot\{\Pi\}$ can be globally defined, not only up to sign.

It will turn out that our symbol $\{\Pi\}$ enjoys the same symmetry properties as the classical 6j symbol (in fact, even more symmetries in some cases). Of course, since $\{\Pi\}$ is only defined up to signs, those symmetries are *a priori* also defined only up to signs. However, by using the resolution of signs eluded above, we are able to make those symmetries precise. Indeed, one of the miracles of the classical 6j symbol, is exactly that the Regge symmetries are valid on the nose, without any unnecessary $-1$s; and we will achieve a similar level of precision in Theorem 5.1.1. It is in fact impossible in our setting to achieve only $+$ signs, but we will have the next best thing and give a rather elegant description of the signs.

All our work elucidating signs comes, however, at a price: one must make a choice of orientation of edges; and in order to obtain the nicest formulas we even need to use a slightly strange one, see the diagram (4.3.1).

## 4. $\mathsf{D}_6$ AND THE TETRAHEDRON

It has been observed by various authors, and perhaps brought into the greatest clarity by Rudenko [Rud22], that the geometry of tetrahedron is related to the root system $\mathsf{D}_6$. As we will soon see, the tetrahedral symbol reflects the geometry of the associated group, and not just its root system.

In the present section, we will set up various notation that will make this connection clearer. For us, the connection between the tetrahedron and $\mathsf{D}_6$ will be " carried" by a homomorphism of rank 6 free abelian groups $\tilde{\mathsf{D}}_6 \hookrightarrow \mathsf{D}_6$ to be defined in § 4.1. In §4.2 we promote this injection of free abelian groups to a homomorphism of compact Lie groups, and in §4.3 we will examine how the Weyl group $W(\mathsf{D}_6)$ interacts with the symmetries of the tetrahedron.

**4.1. The lattice $\mathsf{D}_6$ and its tetrahedral avatar $\tilde{\mathsf{D}}_6$.** The *coroot lattice of type* $\mathsf{D}_n$ will be understood to be the sublattice of $\mathbb{Z}^{2n}$ consisting of elements of the form

$$(x_1, \dots, x_n, -x_n, \dots, -x_1), \quad \sum_{i=1}^{n} x_i \in 2\mathbb{Z}. \tag{4.1.1}$$

Here, and in what follows, a *lattice* is simply a free abelian group of finite rank. It is equipped with the reflection group $W(\mathsf{D}_n) = \{\pm 1\}^{n-1} \rtimes \mathfrak{S}_n$ obtained by permutations of the $x_i$ and changes of an *even* number of signs. It also contains a distinguished $W(\mathsf{D}_n)$-invariant set of *coroots*, namely, all those vectors $\alpha$ that satisfy $(\alpha, \alpha) = 2$ for the standard Euclidean inner product, and the reflections through their orthogonal hyperplanes generate $W$.

Following [Bou02, Plate IV], we identify $\mathsf{D}_n$ with the coweight lattice for the simply-connected group $\mathrm{Spin}_{2n}$ by projecting to the first $n$ coordinates. Here $\mathrm{Spin}_{2n}$ means the universal (two-fold) cover of $\mathrm{SO}_{2n}(\mathbb{C})$; it is a complex semisimple group. The reader who prefers compact Lie groups can equally well work with its maximal compact subgroup, which is similarly described as the the universal (two-fold) cover of the compact group $\mathrm{SO}_{2n}(\mathbb{R})$.

To relate $\mathsf{D}_6$ to the tetrahedron, we consider the set of odd integral-valued functions on oriented edges

$$\tilde{\mathsf{D}}_6 \coloneqq \{f \colon \mathbf{O} \to \mathbb{Z} \mid f(ij) = -f(ji)\},$$

where we say that a function, with domain the oriented edges of the tetrahedron, is *odd* if inverting an edge negates (or inverts, where appropriate) the value of the function. As a convention, we will denote such a function $f$ by means of the $2 \times 3$ matrix of values as follows:

$$\begin{bmatrix} f(\mathbf{12}) & f(\mathbf{13}) & f(\mathbf{14}) \\ f(\mathbf{34}) & f(\mathbf{24}) & f(\mathbf{23}) \end{bmatrix}. \tag{4.1.2}$$

Then $\tilde{\mathsf{D}}_6$ is a lattice of rank 6, and there is an injection

$$\tilde{\mathsf{D}}_6 \longrightarrow \mathsf{D}_6 \tag{4.1.3}$$

which sends a function $f$ to the element of $D_6$ whose twelve coordinates are *sums of values of* $f$ *on opposite edges.* There are three pairs of opposite edges, but each comes with four possible orientations; thus we get twelve such sums in all, symmetric under negation. For definiteness, we take the first six coordinates of the element of $D_6$ to be

$$f(\mathbf{12}) \pm f(\mathbf{34}), \quad f(\mathbf{13}) \pm f(\mathbf{24}), \quad f(\mathbf{14}) \pm f(\mathbf{23}), \tag{4.1.4}$$

in the given order; the last six coordinates are uniquely determined by the first six.

**4.2. The associated isogeny $\tau$ of compact, or reductive, groups.** The isogeny of lattices has a more group-theoretic manifestation which will play an important role in § 8. Namely, there is a commutative diagram

$$\begin{array}{ccc} \tilde{D}_6 \otimes \mathbb{G}_m & \longrightarrow & D_6 \otimes \mathbb{G}_m \\ \downarrow & & \downarrow \\ SL_2^{\mathbf{E}} & \xrightarrow{\tau} & \mathrm{Spin}_{12} \end{array} \tag{4.2.1}$$

where the top row is an isogeny of tori induced from (4.1.3), and the bottom row is a homomorphism of reductive groups in which these tori are maximal.

Here, the notation $(\,\cdot\,) \otimes \mathbb{G}_m$ simply means that we replace integer variables in $(\,\cdot\,)$ by $\mathbb{G}_m$-valued ones; thus, for example, $\tilde{D}_6 \otimes \mathbb{G}_m$ can be considered as functions from oriented edges to $\mathbb{G}_m$ satisfying $f(ij)f(ji) = 1$. Also, as mentioned earlier, the reader who prefers compact groups to reductive groups may harmlessly replace $\mathbb{G}_m$ by the unit circle, $SL_2$ by $SU_2$, and $\mathrm{Spin}_{12}$ by the compact group of the same name.

To construct $\tau$, let $\mathbb{C}^2_{ij}$ be a two-dimensional vector space attached to edge $ij$ with standard basis $\{\mathbf{e}_{ij}, \mathbf{e}_{ji}\}$. Introduce a copy of $SL_2$ indexed by $\{ij\}$, namely, the unimodular automorphisms of $\mathbb{C}^2_{ij}$; its co-character group is then identified with pairs of integers $a_{ij}, a_{ji}$ satisfying $a_{ij} + a_{ji} = 0$. Taking the product over edges $\mathbf{E}$ we arrive at a model for $SL_2^{\mathbf{E}}$ whose maximal torus is canonically identified with $\tilde{D}_6 \otimes \mathbb{G}_m$. Now, for each pair of opposite edges $\{ij\}, \{kl\}$, taking tensor product of the defining representations induces a homomorphism[7]

$$SL_2^{\{ij\}} \times SL_2^{\{kl\}} \longrightarrow SO_4 \subset GL(\mathbb{C}^2_{ij} \otimes \mathbb{C}^2_{kl}),$$

Applying this construction to the three sets of opposite pairs — using the same order that has been used in (4.1.4), that is: $(\mathbf{12}, \mathbf{34})$, $(\mathbf{13}, \mathbf{24})$, $(\mathbf{14}, \mathbf{23})$ — we arrive at a map $SL_2^{\mathbf{E}} \to SO_{12}$, which lifts uniquely to the desired map $\tau : SL_2^{\mathbf{E}} \to \mathrm{Spin}_{12}$ of (4.2.1). Note that $\tau$ isn't an embedding: it has a finite kernel $\{\pm 1\}^2$.

[7]Here, to be precise, $SO_4$ is the *split* form attached to the symmetric pairing in which

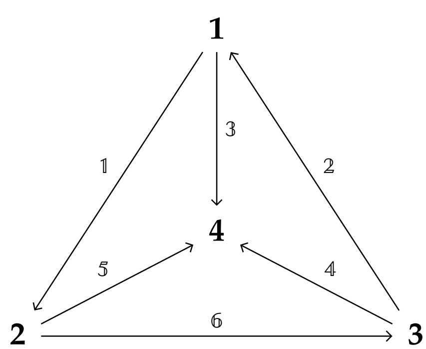
$$\langle \mathbf{e}_{ij} \otimes \mathbf{e}_{kl}, \mathbf{e}_{ji} \otimes \mathbf{e}_{lk} \rangle = 1, \langle \mathbf{e}_{ij} \otimes \mathbf{e}_{lk}, \mathbf{e}_{ji} \otimes \mathbf{e}_{kl} \rangle = -1,$$

and all other pairings zero.

**4.3.** $W(\mathsf{D}_6)$ **and the symmetries of the tetrahedron.** By means of (4.1.4) we understand $W(\mathsf{D}_6)$ to act on $\tilde{\mathsf{D}}_6 \otimes \mathbb{R}$. This action interacts richly with the geometry of the tetrahedron. For example the constraint on dihedral angles is $W(\mathsf{D}_6)$-invariant, see Remark 4.3.4.

We shall now label several important subgroups, which together generate $W(\mathsf{D}_6)$: the orientation reversals, tetrahedral symmetries (which come in two versions: the evident ones, and ones that use an orientation), and the Regge symmetries:

- The group $\mathfrak{I} \simeq (\mathbb{Z}/2)^6$ of orientation reversals: for each $\mathrm{ij} \in \mathbf{O}$ there exists a unique element of $W(\mathsf{D}_6)$ which acts on $\tilde{\mathsf{D}}_6 \otimes \mathbb{R}$ by "reversing the orientation of $\mathrm{ij}$," i.e. negating the value of any function on $\mathrm{ij}$.[8] Such elements generate an elementary abelian subgroup $\mathfrak{I} \subset W(\mathsf{D}_6)$ of order $2^6$.
- The group $\mathfrak{T} \simeq \mathfrak{S}_4$ of tetrahedral symmetries. These arise from physical symmetries of the tetrahedron, acting by permuting vertices in the evident fashion; for example, $(12) \in \mathfrak{S}_4$ will *negate* the value of $\mathrm{f} \in \tilde{\mathsf{D}}_6 \otimes \mathbb{R}$ at **12**.
- The *subset* $\mathfrak{R} = \{\mathrm{r}_{\mathbb{14}}, \mathrm{r}_{\mathbb{25}}, \mathrm{r}_{\mathbb{36}}\}$ of *Regge symmetries*, indexed by pairs of opposite edges. These "exotic" symmetries were written down by Regge; they do not form a group, *and they depend on choice of an orientation*, that is to say, a splitting $\mathbf{E} \to \mathbf{O}$.
  - To elegantly resolve an important sign ambiguity down the line (see § 3.6 and Theorem 5.1.1), we will choose not the "dictionary" order, i.e. coming from the order on the natural numbers $\mathbf{1} < \mathbf{2} < \mathbf{3} < \mathbf{4}$, but rather the "vortex"orientation[9] defined by the following diagram (4.3.1), where we also use blackboard bold numbers $\mathbb{1}$, etc. to denote each oriented edge.[10]




(4.3.1)

  - Note that the orientation is almost according to the dictionary order in $\mathbf{V}$, with one exception: the edge **31** is favored over **13**, making the outer triangle an oriented cycle rather than a simplex.
  - For short we write $\mathrm{r}_{\mathbb{14}}$ for the element associated to $\{\mathbf{12}, \mathbf{34}\}$, which is defined as follows: For $\mathrm{f} \in \tilde{\mathsf{D}}_6 \otimes \mathbb{R}$, let $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$ be its values at $\mathbf{31}, \mathbf{14}, \mathbf{23}, \mathbf{24}$, i.e. the remaining four edges, oriented according to the chosen orientation. Then

[8]These *should not be confused* with the elements of $W(\mathsf{D}_6)$ that switch the signs of some coordinates $x_i$.

[9]There are multiple choices possible, but they are not completely random and the "admissible" choices all lead to the same end.

[10]We do apologize in advance for using different font shapes maybe a little excessively.

$r_{\mathbb{14}}(f)$ has the same values at **12** and **34**, but $a, b, c, d$ are modified according to

$$a^* = s - a, \quad b^* = s - b, \quad c^* = s - c, \quad d^* = s - d,$$

where $s := \frac{a+b+c+d}{2}$ is the "semi-perimeter." We proceed similarly for $r_{\mathbb{25}}$ and $r_{\mathbb{36}}$.

- For technical use only: the group $\mathfrak{T}^{\mathrm{or}} \simeq \mathfrak{S}_4$ of oriented tetrahedral symmetries. These also arise from physical symmetries of the tetrahedron, but *taking account of orientation.* We identify $\tilde{\mathsf{D}}_6$ with functions on the set **E** using the vortex orientation of (4.3.1), and then $\mathfrak{T}^{\mathrm{or}}$ acts in the natural way on **E**. Thus, for example, the transposition $(12) \in \mathfrak{S}_4$ acting on $f \in \tilde{\mathsf{D}}_6$ leaves its value at **12** unchanged. We call $\mathfrak{T}^{\mathrm{or}}$ the *oriented tetrahedral group.* .

To avoid potential confusion when compared with notations such as (4.1.2), we emphasize again that

> *the vortex orientation is to be used only when we consider the symmetries $\mathfrak{T}^{\mathrm{or}}$ and $\mathfrak{R}$, particularly the subtle sign issue associated with them. By default, in all other instances unless specified otherwise, we use the ordering orientation.*

We now discuss some of the group theory of how these various groups interact. For the moment, the reader can skip this discussion, and refer back to it as necessary:

**Lemma 4.3.1.** *Together with the group of orientation reversals $\mathfrak{I}$, either the group of tetrahedral symmetries $\mathfrak{T}$ or the group of oriented tetrahedral symmetries $\mathfrak{T}^{\mathrm{or}}$ generate the same subgroup of $W(\mathsf{D}_6)$; this subgroup is isomorphic to $(\mathbb{Z}/2)^6 \rtimes \mathfrak{S}_4$.*

**Lemma 4.3.2.** *Suppose we put the oriented edges in the following matrix:*

$$\begin{bmatrix} \mathbf{12} & \mathbf{31} & \mathbf{14} \\ \mathbf{34} & \mathbf{24} & \mathbf{23} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \mathbb{2} & \mathbb{3} \\ \mathbb{4} & \mathbb{5} & \mathbb{6} \end{bmatrix},$$

*and let*

(1) $r_i$ *be the Regge symmetry that fixes the* $i$*-th column (so that* $r_1 = r_{\mathbb{14}}$*, and so on);*
(2) $h_i \in \mathfrak{T}^{\mathrm{or}}$ *fix the* $i$*-th column and swaps the two rows in the other columns;*
(3) $\nu_i \in \mathfrak{T}^{\mathrm{or}}$ *fix the* $i$*-th column and swap the two columns other than the* $i$*-th.*

*Then the group generated by $\mathfrak{T}^{\mathrm{or}}$ and $\mathfrak{R}$ is isomorphic to $\mathfrak{S}_3 \times \mathfrak{S}_4$, with $\mathfrak{S}_3$ generated by $\{h_i r_i = r_i h_i\}$, and the commuting $\mathfrak{S}_4$ generated by $\{\nu_i r_i = r_i \nu_i\}$.*

*Proof.* Relabel coordinates $x_1, \ldots, x_6$ in $\mathsf{D}_6$ (cf. (4.1.1)) as $y_1^\pm, y_2^\pm, y_3^\pm$, so that $y_1^\pm$ correspond to $\mathbf{12} \pm \mathbf{34}$, $y_2^\pm$ correspond to $\mathbf{31} \pm \mathbf{24}$, and so on. Then $r_1, h_1, \nu_1$ act as follows: $r_1$ swaps $y_2^+$ with $y_3^+$ and negates $y_2^-, y_3^-$; $h_1$ negates $y_2^-$ and $y_3^-$, and $\nu_1$ swaps $y_2^\pm$ with $y_3^\pm$. Then $\nu_1 r_1$ (resp. $h_1 r_1$) fixes four of the coordinates and acts as a reflection in the remaining two:

$$\begin{bmatrix} y_1^+ & y_2^+ & y_3^+ \\ y_1^- & y_2^- & y_3^- \end{bmatrix} \longmapsto \begin{bmatrix} y_1^+ & y_2^+ & y_3^+ \\ y_1^- & -y_3^- & -y_2^- \end{bmatrix} \text{ resp. } \begin{bmatrix} y_1^+ & y_3^+ & y_2^+ \\ y_1^- & y_2^- & y_3^- \end{bmatrix}.$$

The action of the other $\nu_i r_i$s and $h_i r_i$s is similar. The result is now an easy exercise. ∎

**Lemma 4.3.3.** *Any of the following collections generate all of $W(\mathsf{D}_6)$:*

(1) *the group* $\mathfrak{I}$ *and subset* $\mathfrak{R}$*;*
(2) *the groups* $\mathfrak{I}$*,* $\mathfrak{T}^{\mathrm{or}}$*, and any single element in* $\mathfrak{R}$*;*
(3) *the groups* $\mathfrak{I}$*,* $\mathfrak{T}$*, and any single element in* $\mathfrak{R}$*.*

*Proof.* We use the notation in the proof of Lemma 4.3.2. Considering the permutation action on the $y^{\pm}$s and ignoring signs gives a map $W(\mathsf{D}_6) \longrightarrow \mathfrak{S}_6$. We will first of all describe the images of $\mathfrak{I}, \mathfrak{R}, \mathfrak{T}^{\mathrm{or}}$ under this map:

- Orientation reversal symmetries have image in $\mathfrak{S}_6$ given by a transposition that switches $y_i^+$ with $y_i^-$ for one value of $i$. Consequently, the image of of $\mathfrak{I}$ in $\mathfrak{S}_6$ is isomorphic to $(\mathbb{Z}/2)^3$.
- Regge symmetries in $\mathfrak{R}$ induce transpositions on the $y^+$s while fixing all $y^-$ coordinates.
- Oriented tetrahedral symmetries in $\mathfrak{T}^{\mathrm{or}}$: these stabilize the $y^+$s and $y^-$s and act on them according to the *same* permutation of the index set $1, 2, 3$.

Conjugating orientation reversal by Regge symmetries, we can produce any transposition swapping any one of $\{y_1^+, y_2^+, y_3^+\}$ with any one of $\{y_1^-, y_2^-, y_3^-\}$, and then further conjugating by orientation symmetries, we can produce all transpositions. So, the group generated by $\mathfrak{I}$ and $\mathfrak{R}$ surjects onto $\mathfrak{S}_6$, with kernel a permutation-invariant subgroup of $(\mathbb{Z}/2)_0^6$ (the subscript means elements of sum zero). The order of this kernel is at least $8$, since $\mathfrak{I}$ has order $2^6$ but its image in $\mathfrak{S}_6$ has order $2^3$; therefore, this kernel is all of $(\mathbb{Z}/2)_0^6$.

Clearly $\mathfrak{R}$ is contained in the group generated by a single Regge symmetry and $\mathfrak{T}^{\mathrm{or}}$, so the second claim follows. The third claim then follows from Lemma 4.3.1. ∎

*Remark* 4.3.4. Here is a manifestation of $W(\mathsf{D}_6)$ in the geometry of Euclidean tetrahedra. For such a tetrahedron, choose an odd function $\theta_{ij}$ on oriented edges so that $|\theta_{ij}|$ gives the $ij$ dihedral angle and put

$$\mathbf{x} = \text{image of } e^{i\theta_{ij}} \in \tilde{\mathsf{D}}_6 \otimes \mathbb{C}^\times \text{ inside } \mathsf{D}_6 \otimes \mathbb{C}^\times.$$

Then [PF20, § 3, (2)] says precisely that the constraint to form the dihedral angles of a Euclidean tetrahedron takes the form

$$P \equiv 0$$

where $P$ is a $W(\mathsf{D}_6)$-*invariant* regular function on the torus $\mathsf{D}_6 \otimes \mathbb{C}^\times$. In fact, $\mathsf{D}_6 \otimes \mathbb{C}^\times$ is the maximal torus of the group $\mathrm{Spin}_{12}$, and in this way we can regard $\mathbf{x}$ as an element of the compact group of this type. With this identification, the polynomial $P$ is given by a linear combination of the characters of the trivial, half-spin, adjoint, and symmetric square representations.

**4.4. The "vertex" and "face" half-spin representations.** The spin representation of $\mathrm{Spin}_{12}$, and its relation to the geometry of the tetrahedron through the prior discussion, will be particularly relevant for us later on, and we set up notation now.

Recall that, given an element of $\mathrm{Spin}_{2n}$ whose image in $\mathrm{SO}_{2n}$ has standard eigenvalues $\lambda_1^{\pm}, \ldots, \lambda_n^{\pm}$, its eigenvalues in the two half-spin representations are collectively given by

$$\prod_{i=1}^{n} \lambda_i^{\pm\frac{1}{2}}.$$

One half-spin involves those eigenvalues with an *even* number of $-1$ signs, and the other half-spin involves the remaining eigenvalues, i.e., those with an *odd* number of $-1$ signs.

In the case of $\mathrm{Spin}_{12}$, both the half-spin representations are of dimension 32. Each half-spin gives rise to a set of weights in the dual of $\mathsf{D}_6$, and so also, by means of $\tilde{\mathsf{D}}_6 \to \mathsf{D}_6$, to a set of 32 weights in the dual of $\tilde{\mathsf{D}}_6$. For one of the half-spins, which we call S, this set of weights consists of all functionals

$$f \mapsto \pm f(ij) \pm f(ik) \pm f(il),$$

that is, where we sum f over three edges that meet at a vertex. For the other half-spin, the set of weights is similarly defined but now involving three edges that span a face. We will be interested in the first, or "vertex", half-spin representation S. Note that:

- The pullback of S via $\tau : \mathrm{SL}_2^{\mathbf{E}} \to \mathrm{Spin}_{12}$ is given by the sum of four copies of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, where we tensor together the standard representations for each triple of edges that meet in a common vertex.
- If we choose $\mathbf{x} \in \tilde{\mathsf{D}}_6 \otimes \mathbb{G}_m$, specified by a set of coordinates $x_{ij}$ satisfying $x_{ij}x_{ji} = 1$, the eigenvalues of the corresponding toral elements on S are the 32 products

$$x_{ij}^{\pm} x_{ik}^{\pm} x_{il}^{\pm}, \tag{4.4.1}$$

  where $\{ij, ik, il\} = \mathbf{O}_i$ are edges sharing one vertex $i$.

There is an important splitting into a direct sum of Lagrangian subspaces

$$S = S^+ \oplus S^-,$$

stable under the diagonal torus of $\mathrm{SL}_2^{\mathbf{E}}$. Namely, we take $S^+$ to contain all eigenspaces arising from eigenvalues (4.4.1) where there are at least two $+$ signs amongst the exponents; similarly, we define $S^-$ to correspond to eigenvalues where there are at least two negative signs amongst the exponents.

## 5. The main theorems

In this section, we formulate the two main results of the present paper. The first (Theorem 5.1.1) shows that $\{\Pi\}$ enjoy a $W(\mathsf{D}_6)$-symmetry in the principal series case. The second (Theorem 5.2.1) gives an explicit evaluation when $\Pi$ is additionally unramified. Both theorems give explicit ways, within the contexts to which they apply, of resolving the sign ambiguity of $\{\Pi\}$.

**5.1. Weyl symmetry for principal series.** Let us assume $\mathcal{R}$ is split and consider now the tetrahedral symbol on principal series representations. As we will see, in this situation, we will be able to resolve the sign ambiguity and also will find a symmetry by the Weyl group of $\mathsf{D}_6$.

We will first fix a good parameterization of such representations by character-valued *odd* functions on oriented edges:

$$\mathcal{X} := \{\chi_{ij} \in \widehat{\mathsf{F}^\times} \mid \chi_{ij}\chi_{ji} = 1\},$$

where $\widehat{\mathsf{F}^\times}$ denotes the set of characters (i.e., unitary quasi-characters) of $\mathsf{F}^\times$. As observed after (4.2.1), the space $\mathcal{X}$ can be identified with the *tensor product of abelian groups*:

$$\begin{gathered} \tilde{\mathsf{D}}_6 \otimes \widehat{\mathsf{F}^\times} \xrightarrow{\sim} \mathcal{X}, \\ \mathsf{f} \otimes \lambda \longmapsto \chi_{ij} = \lambda^{\mathsf{f}(ij)} \end{gathered} \tag{5.1.1}$$

For such $\chi_{ij}$, let $\pi_{ij}$ be the principal series attached to $\chi_{ij}$, as has been defined in § 3.5.1; note that the isomorphism class of $\pi_{ij}$ does not depend on the order of $i$ and $j$. By analogy with the standard notation for classical 6j symbols, we shall write

$$\begin{Bmatrix} \chi_{12} & \chi_{13} & \chi_{14} \\ \chi_{34} & \chi_{24} & \chi_{23} \end{Bmatrix} = \{\chi\}$$

for $\{\Pi\}$ in this case. It turns out that $\{\Pi\}$ is not identically zero on any components where all $\pi$ are principal series. By Proposition 3.5.3, the square of this symbol extends from $\mathcal{X}$ to a meromorphic function on the space $\mathcal{X}_{\mathbb{C}}$ of odd functions from oriented edges to quasi-characters.

In order to make better sense of this, we will need to consider the map

$$\tilde{\mathsf{D}}_6 \otimes \widehat{\mathsf{F}^\times} \longrightarrow \mathsf{D}_6 \otimes \widehat{\mathsf{F}^\times}. \tag{5.1.2}$$

induced by $\tilde{\mathsf{D}}_6 \to \mathsf{D}_6$ from (4.1.4). The right hand side can be considered as collections of characters $\psi_1, \dots, \psi_6$, *together with a chosen square root of the product* $\psi_1 \cdots \psi_6$; the morphism (5.1.2) sends $\chi_{ij}$ to the $\psi$s given by $\chi_{12}\chi_{34}^{\pm}, \chi_{13}\chi_{24}^{\pm}, \chi_{14}\chi_{23}^{\pm}$ together with the square root of their product given by $\chi_{12}\chi_{13}\chi_{14}$. In particular, all products $\chi_{ij}^{\pm}\chi_{ik}^{\pm}\chi_{il}^{\pm}$, as well as all products of the form $\chi_{ij}^{\pm}\chi_{jk}^{\pm}\chi_{ki}^{\pm}$, depend only on the image of $\chi_{ij}$ under (5.1.2).

To efficiently state our main results, we introduce the following set of 32 characters:

$$\mathsf{S} = \left\{\chi_{ij}^{\pm}\chi_{ik}^{\pm}\chi_{il}^{\pm} \,\middle|\, i \in \mathbf{V}\right\}. \tag{5.1.3}$$

where for each of the four $i \in \mathbf{V}$, we allow all possible choices of the three signs. This $\mathsf{S}$ is visibly an avatar of the weights of the spin representation introduced in § 4.4; the connection will be made even clearer in § 9.5. We divide $\mathsf{S} = \mathsf{S}^+ \coprod \mathsf{S}^-$, where $\mathsf{S}^+ \subset \mathsf{S}$ contains all terms that involve either three $+1$'s in the exponents, or two $+1$'s and one $-1$, whereas $\mathsf{S}^- = \mathsf{S} - \mathsf{S}^+$. Write $\gamma(s, \mathsf{S}), \gamma(s, \mathsf{S}^+), \gamma(s, \mathsf{S}^-)$ for the corresponding $\gamma$-functions, where we follow the notation of § 2.2.5 and define $\gamma(s, -)$ for a multiset to be the product of the constituent $\gamma$-functions; we have

$$\gamma(s, \mathsf{S}) = \gamma(s, \mathsf{S}^+)\gamma(s, \mathsf{S}^-).$$

The following is our first main theorem:

**Theorem 5.1.1.** *We have the following results:*

(1) *The value of* $\{\chi\}^2$ *depends only on three-fold products* $\chi^{\pm}_{ij}\chi^{\pm}_{ik}\chi^{\pm}_{il}$ *and in particular the rule* $\chi \mapsto \{\chi\}^2$ *factors through the map*

$$\mathfrak{X} \simeq \tilde{\mathsf{D}}_6 \otimes \widehat{\mathsf{F}^\times} \longrightarrow \mathsf{D}_6 \otimes \widehat{\mathsf{F}^\times}, \tag{5.1.4}$$

*in such a way that the extended function on the right hand side is invariant by the action of* $W = W(\mathsf{D}_6)$ *(cf. § 4.1 for the* $W$*-action).*

(2) *More precisely, the rule*

$$\chi \longmapsto \begin{Bmatrix} \chi_{12} & \chi_{13} & \chi_{14} \\ \chi_{34} & \chi_{24} & \chi_{23} \end{Bmatrix} \sqrt{\gamma\Big(\frac{1}{2}, S^-\Big)}$$

*can be globally defined on* $\mathsf{D}_6 \otimes \widehat{\mathsf{F}^\times}$*: there exists a meromorphic function* $\mathrm{I}$ *defined on* $\mathsf{D}_6 \otimes \widehat{\mathsf{F}^\times}$ *that agrees with the right-hand side for suitable choice of sign of* $\sqrt{\gamma}$*. This function can be chosen to depend only on products of the form* $\chi_{ij}\chi_{ik}\chi_{il}^{-1}$ *and satisfy the following* $W$*-equivariance:*

$$\frac{\mathrm{I}(w^{-1}\chi)}{\mathrm{I}(\chi)} = \iota(w,\chi)\gamma\Big(\frac{1}{2}, S^+ \cap w(S^-)\Big), \tag{5.1.5}$$

*where* $\iota(w,\chi) \in \pm 1$ *is characterized by the fact that the right hand side of* (5.1.5) *is a* 1*-cocycle of* $W$ *valued in the multiplicative group of nonzero meromorphic functions on* $\mathsf{D}_6 \otimes \widehat{\mathsf{F}^\times}$*, and the following facts, using notation as in § 4.3:*

(a) *when* $w \in \mathfrak{T}$ *is a tetrahedral symmetry or* $w \in \mathfrak{R}$ *is a Regge symmetry,* $\iota(w,\chi) = 1$*;*

(b) *when* $w$ *is the element* $s_{ij}$ *that exchanges* $\chi_{ij}$ *with* $\chi_{ji}$*,* $\iota(w,\chi) = \chi_{ik}\chi_{il}\chi_{jk}\chi_{jl}(-1)$*, where* $\{i,j,k,l\} = \mathbf{V}$*.*

A proof is sketched in § 5.3. The proof proper appears in § 12.3.

*Remark* 5.1.2. (1) We note that the extension of $\{\chi\}$ to $\mathsf{D}_6 \otimes \widehat{\mathsf{F}^\times}$ is no longer the tetrahedral symbol for $\mathcal{R}$, because (5.1.4) is not surjective. It would be interesting to interpret the extension in terms of tetrahedral symbols for a spin group rather than $\mathcal{R}$, i.e., in the noncompact case, $\mathrm{SL}_2$ rather than $\mathrm{PGL}_2$, cf. § 9.4.

(2) It is not a formality that a choice of signs for $\iota(w,\chi)$ exists. It is equivalent to the following fact: the 2-cocycle on $W$ defined by

$$(w,w') \longmapsto \Bigg(\chi \mapsto \prod_{\substack{\psi \in S^+, w^{-1}\psi \in S^- \\ (ww')^{-1}\psi \in S^+}} \psi(-1)\Bigg)$$

is cohomologically trivial.

(3) We will define the function I using a certain hypergeometric formula for the tetrahedral symbol (see §§ 6.3 and 7).

(4) The statement above does not *directly* cover Regge's original symmetry of 6j symbols, which pertains to the case of $\mathcal{R}$ *compact*. We outline our expectations about this in § 9.5.4.

**5.2. Evaluation of $\{\Pi\}$ in the unramified case.** Suppose now that $\mathsf{F}$ is nonarchimedean, and let $\mathbb{T} \subset \mathbb{C}^\times$ be the unit circle. There is an embedding of groups

$$\begin{aligned} \mathbb{T} &\longrightarrow \widehat{\mathsf{F}^\times} \\ z &\longmapsto (x \mapsto z^{\mathrm{val}_\mathsf{F}(x)}), \end{aligned}$$

which, in words, sends a complex number $z$ of absolute value 1 to the character of $\mathsf{F}^\times$ that sends a uniformizer to $z$. The image of this homomorphism gives the unitary unramified characters of $\mathsf{F}^\times$, i.e., those that are trivial on the maximal compact subgroup $\mathcal{O}^\times \subset \mathsf{F}^\times$. The isomorphism (5.1.1) identifies the subspace $\mathcal{X}_0 \subset \mathcal{X}$ consisting of unramified unitary characters with the tensor product

$$\mathcal{X}_0 \simeq \tilde{\mathsf{D}}_6 \otimes \mathbb{T}, \tag{5.2.1}$$

and by Theorem 5.1.1, the function $\mathsf{J}_\chi$ descends to a $W(\mathsf{D}_6)$-invariant function on

$$\mathsf{D}_6 \otimes \mathbb{T} = \text{the maximal torus of } \mathrm{Spin}_{12},$$

where we now understand $\mathrm{Spin}_{12}$ to mean the compact group of that type, and the identification was that discussed after (4.1.1).

Now, the $W(\mathsf{D}_6)$-invariant function on $\mathsf{D}_6 \otimes \mathbb{T}$ are precisely those arising by restriction of class functions (i.e., conjugacy-invariant continuous functions) on $\mathrm{Spin}_{12}$. It is therefore, we hope, irresistible to ask for a description of this class function in terms of the representation theory of $\mathrm{Spin}_{12}$.

As discussed in § 4.4, there is a 32-dimensional half-spin representation $\mathsf{S}$ of $\mathrm{Spin}_{12}$ whose weights pull back under $\tilde{\mathsf{D}}_6 \to \mathsf{D}_6$ to the 32 linear functionals

$$\{\pm \mathsf{f}(ij) \pm \mathsf{f}(ik) \pm \mathsf{f}(il)\}$$

where $\{ij, ik, il\} = \mathbf{O}_i$ are edges sharing starting vertex $i$. It contains a distinguished 16-dimensional *cone of pure spinors* $\mathsf{P} \subset \mathsf{S}$ (for details see § 13.2) viewed as a complex subvariety of $\mathsf{S}$. Its ring of regular functions $\mathbb{C}[\mathsf{P}]$ then affords a weighted $\mathbb{C}^\times = \mathbb{G}_m(\mathbb{C})$-action induced by the scaling on $\mathsf{S}$.

**Theorem 5.2.1.** *Suppose* $\mathsf{F}$ *is nonarchimedean and* $\chi \in \mathcal{X}_0$ *with image* $\sigma \in \mathsf{D}_6 \otimes \mathbb{T}$. *Then*

$$\begin{Bmatrix} \chi_{12} & \chi_{13} & \chi_{14} \\ \chi_{34} & \chi_{24} & \chi_{23} \end{Bmatrix} = \frac{(1-q^{-2})^3}{\sqrt{L(\frac{1}{2}, \mathsf{S})}} \operatorname{Tr}\big(q^{-\frac{1}{2}}\sigma, \mathbb{C}[\mathsf{P}]\big), \tag{5.2.2}$$

*where* $q^{-\frac{1}{2}}$ *acts on* $\mathbb{C}[\mathsf{P}]$ *through the weighted action of* $\mathbb{C}^\times$, *and* $\mathsf{S}$ *is as in* (5.1.3).

A proof is sketched in § 5.3 and detailed in § 13.

In words, up to normalizing factors (see § 15.2 for a geometric point of view concerning the factor $(1-q^{-2})^3$), the tetrahedral symbol is *the weighted character of the Frobenius action on the algebraic function ring of the spinor cone.* Why are the half-spin representation and the spinor cone involved? Relative Langlands duality offers at least a context in which to understand this, explaining, for example, why the spinor cone is actually Lagrangian inside the half-spin; see § 8 for discussion.

**5.3. A sketch of the proofs of Theorems 5.1.1 and 5.2.1.** In § 6.3, we will write down an explicit integral formula (the "edge formula") for $\{\chi\}$, whose proof is in § 11. The integral is then made more explicit in § 7 into a *hypergeometric* form. It will be clear by then — for example, from (7.2.2) — that $\{\chi\}^2$ depends only on characters of the form $\chi^{\pm}_{ij}\chi^{\pm}_{ik}\chi^{\pm}_{il}$. This proves (5.1.4).

For the $W(\mathsf{D}_6)$-symmetry, we construct explicit symmetries of $\{\chi\}$ that generate the whole Weyl group, corresponding to the generators discussed in § 4.3. The least trivial ones are the Regge symmetries, which come from a Fourier duality result for hypergeometric integrals. This proves the symmetry up to signs. The signs can then be pinned down using the Mellin transform of the hypergeometric integral from § 14.6. See § 12 for the details.

Now consider Theorem 5.2.1. The right-hand side of (5.2.2) can be computed using the Weyl character formula after we decompose $\mathbb{C}[\mathsf{P}]$ into irreducible $\mathrm{Spin}_{12}$-representations. To analyze the left-hand side we rely on another integral formula (the "vertex formula")[11] for $\{\chi\}$ that we will prove in § 6.2; it leads to a computation involving the Bruhat–Tits tree of $\mathrm{PGL}_2(F)$. A direct comparison of both sides is thus conceptually possible yet seems a bit tedious to do even with a computer. We will instead compare both sides in these steps:

(1) Showing that both sides are meromorphic with the same set of simple poles;
(2) Assisted by a computer, showing that their residues agree (which is significantly easier than comparing the whole expressions);
(3) Showing that the difference of both sides is a bounded function with value $0$ at a single input, so it must be identically $0$.

For details, see § 13.

## 6. Integral formulas for the tetrahedral symbol

In this section we will give a class of integral formulas for the tetrahedral symbol. They are based on the interaction between tetrahedral combinatorics and the geometry of certain $\mathcal{R}$-spaces.

They come in two classes, which can be seen as dual to one another: a *vertex integral* which takes as input an $\mathcal{R}$-space $X$ and a collection of $\mathcal{R}$-invariant functions $\varphi_{ij} : X^2 \to \mathbb{C}$ indexed by (unoriented) edges, and an *edge integral* which takes as input a $\mathcal{R}$-space $Y$ and a collection of $\mathcal{R}$-invariant functions $\psi_i : Y^3 \to \mathbb{C}$ indexed by vertices. These are given, respectively, by

$$I^{\mathbf{V}}(X, \varphi) := \int_{\mathcal{R}\backslash X^{\mathbf{V}}} \prod_{ij\in \mathbf{E}} \varphi_{ij}(x_i, x_j), \text{ and } I^{\mathbf{E}}(Y, \psi) := \int_{\mathcal{R}\backslash Y^{\mathbf{E}}} \prod_{i\in \mathbf{V}} \psi_i(y_{ij}, y_{ik}, y_{il}).$$

In both forms, the group $\mathcal{R}$ acts on $X^{\mathbf{V}}$ or $Y^{\mathbf{E}}$ diagonally, and the integrands are $\mathcal{R}$-invariant. The symmetry between the two constructions is a little clearer if we describe the situation in words: To each $X$-labelling of vertices, i.e. "tetrahedra in $X$," we can attach a number, namely, the product over edges $e$, of the values of $\varphi_e$ on vertices incident with

[11] In principle we might try to use the hypergeometric formula as well, but but the vertex formula seems to be easier to work with for this purpose.

$e$. We obtain $I^{\mathbf{V}}$ by integrating this number over the space of X-labellings of vertices. For $I^{\mathbf{E}}$ we just switch X with Y, vertices with edges, and $\varphi$ with $\psi$.

**6.1. Vertex formula for $\mathcal{R}$ compact.** The following vertex formula, valid for $\mathcal{R}$ compact, was already given by Wigner.

**Proposition 6.1.1** (Wigner). *Suppose that $\mathcal{R}$ is compact. Then $\{\Pi\}^2$ is the vertex integral $I^{\mathbf{V}}(X, \varphi)$ associated to $X = \mathcal{R}$ and $\varphi_{ij}(g_i, g_j)$ the character of $\pi_{ij}$ evaluated at $g_i g_j^{-1}$:*

$$\{\Pi\}^2 = \int_{\mathcal{R}^{\mathbf{V}}} \prod_{ij \in \mathbf{E}} \mathrm{Tr} \circ \pi_{ij}(g_i g_j^{-1}).$$

*Proof.* We follow the notation of § 3.4.1. Let $\delta \in \Pi_G$ represent the contraction mapping, so that $(v, \delta) = \langle v \rangle$ for all $v \in V$. By definition, $\{\Pi\} = (V, \delta)$. Now, for any vector $v, w$ we have

$$(V, v)(V, w) = \int_{h \in H} (hv, w),$$

for both sides determine $H \times H$-invariant functionals on $\Pi_G \otimes \Pi_G$ with the same value 1 at $V \otimes V$. Substitute $v = w = \delta$ to find:

$$\{\Pi\}^2 = \int_H (h\delta, \delta).$$

However, Lemma 6.1.2 below implies that for $h = (g_i)_{i \in \mathbf{V}}$, we have

$$(h\delta, \delta) = \prod_{ij \in \mathbf{E}} \mathrm{Tr} \circ \pi_{ij}(g_i g_j^{-1}),$$

which readily implies the desired formula. ∎

**Lemma 6.1.2.** *Suppose W is a finite-dimensional complex vector space equipped with a nondegenerate symmetric pairing $(-,-)$. Let $\delta \in W \otimes W$ represent the pairing, so that $(\delta, v \otimes w) = (v, w)$. Then*

$$(A\delta, \delta) = \mathrm{Tr}(A)$$

*for any endomorphism $A$ of W.*

*Proof.* Fix an orthonormal basis $e_k$ with respect to the self-pairing $(-,-)$. Then $\delta = \sum_k e_k \otimes e_k$ and

$$(A\delta, \delta) = \sum_k (Ae_k, e_k) = \mathrm{Tr}(A),$$

as claimed. ∎

**6.2. Vertex formula for $F$ nonarchimedean and $\pi$ unramified.** Suppose now that $\mathcal{R}$ is noncompact and the $\pi_{ij}$ are *unramified* principal series (see § 5.2) induced from $\chi_{ij}$. In this case, each $\pi_{ij}$ admits a one-dimensional space of vectors invariant under the maximal compact subgroup

$$K \subset \mathcal{R}.$$

Explicitly, in the non-archimedean case, with reference to an isomorphism $\mathcal{R} \simeq \mathrm{PGL}_2(F)$, we have $K \simeq \mathrm{PGL}_2(\mathcal{O})$. We will henceforth assume that $F$ is nonarchimedean; the discussion below goes over in the archimedean case with minor modifications.

We shall describe a class of vertex formulas, valid for such representations. Take $X$ to be $\mathcal{R}/K$ where $K$ is as above. Fix $v_{ij} \in \pi_{ij}$ a nonzero $K$-fixed vector normalized so that $(v_{ij}, v_{ij}) = 1$; this is possible because, again, the symmetric pairing is positive definite on a real structure. Therefore $v_{ij}$ is uniquely specified up to sign. Now define $\varphi_{ij} = X \times X \longrightarrow \mathbb{C}$ by the rule

$$\varphi_{ij}(g_iK, g_jK) = (g_iv_{ij}, g_jv_{ij}).$$

This is the "spherical function for $\pi_{ij}$".

**Proposition 6.2.1.** *With the notations above, we have (up to sign)*

$$\begin{Bmatrix} \chi_{12} & \chi_{13} & \chi_{14} \\ \chi_{34} & \chi_{24} & \chi_{23} \end{Bmatrix} = \frac{L(1, \mathrm{ad})}{L(2)^8 \sqrt{L(\frac{1}{2}, S)}} \cdot \mathrm{I}^{\mathbf{V}}(X, \varphi)$$

$$\left( = \frac{L(1, \mathrm{ad})}{L(2)^8 \sqrt{L(\frac{1}{2}, S)}} \cdot \int_{g_i \in \mathcal{R}^{\mathbf{V}}} \prod_{ij \in \mathbf{E}} (g_iv_{ij}, g_jv_{ij}) \right),$$

*where, on the right-hand side, we take counting measure on the discrete set* $X$, *and put on* $\mathcal{R}$ *the measure for which the volume of* $K$ *equals* $1$, $S$ *is as in § 5.1, and*

$$L(1, \mathrm{ad}) = \prod_{ij \in \mathbf{E}} L(1)L(1, \chi_{ij}^2)L(1, \chi_{ij}^{-2}),$$

*and recall the convention that* $L(s)$ *is the value of the* $L$*-function for the trivial character at* $s$.

*Remark* 6.2.2. In words, Proposition 6.2.1 asserts that (up to the normalizing factors) we obtain the tetrahedral symbol by integrating over "moduli of tetrahedra in X" the product of spherical functions labeled by the edges. Note that it implies that there is a coherent sign choice for the *normalized* tetrahedral symbol when the $\pi$s vary through unramified representations.

*Proof of Proposition 6.2.1.* Again we follow the notation of § 3.4.1. Let

$$v = \bigotimes_{ij \in \mathbf{E}} (v_{ij} \otimes v_{ij}) \in \Pi_G.$$

We shall compute separately $\Lambda^H(v)$ and $(\Lambda^H)'(v)$. By definition,

$$(\Lambda^H)'(v) = \int_{\mathcal{R} \backslash H} (g_iv_{ij}, g_jv_{ji}) = \int_{\mathcal{R} \backslash X^{\mathbf{V}}} \prod_{ij \in \mathbf{E}} \varphi_{ij}(x_i, x_j) = \mathrm{I}^{\mathbf{V}}(\mathcal{R}/K, \varphi).$$

On the other hand, $\Lambda^H(v)$ must be evaluated by hand. It is known (a special case of [II10, Theorem 1.2], or an easy if rather tedious computation in the case at hand) that

$$\Lambda^H(v) = L(2)^8 \frac{\sqrt{L(\frac{1}{2}, S)}}{L(1, \mathrm{ad})}. \tag{6.2.1}$$

Comparing this with the definition $(\Lambda^{H})' = \{\Pi\}\Lambda^{H}$, we arrive at the claimed formula. ∎

The same reasoning also works in the case of $F$ archimedean, but we do not know a reference for the computation (6.2.1) there.

**6.3. Edge formula for principal series.** The edge formula will be applicable to the case of the *split* $\mathcal{R}$ and the $\pi_{ij}$ principal series induced by character $\chi_{ij}$, for which we retain the notations from § 5.1.

We take $Y = \mathbb{P}^1_F$. The $\psi_i$ will not be functions on $Y \times Y \times Y$ itself, but rather sections of certain line bundles — they will be functions on $F^2 \times F^2 \times F^2$, with certain degrees of homogeneity in each factor. Their *product* will define a $(-2)^{\mathbf{E}}$-homogeneous function on $(F^2)^{\mathbf{E}}$, i.e. a *density* — this can be integrated over $Y^{\mathbf{E}} \simeq (\mathbb{P}^1_F)^{\mathbf{E}}$ as explained in § 2.4.

The functions $\psi_i$ are defined (as sections of suitable bundles) on that open subset of $Y^{\mathbf{E}}$ where adjacent edges are assigned distinct labels (in Y), as follows:

$$\psi_i = \psi_i^j \psi_i^k \psi_i^l,$$

where $\{i, j, k, l\} = \mathbf{V}$ and

$$\psi_i^l \coloneqq \chi_{i-}^l (y_{ij} \wedge y_{ik}) = (\chi_i^l |\cdot|^{-\frac{1}{2}})(y_{ij} \wedge y_{ik}), \text{ and } \chi_i^l \coloneqq \frac{\chi_{ij}\chi_{ik}}{\chi_{il}}.$$

Here we write $x \wedge y$, for $x, y \in F^2$, for the determinant of the $2 \times 2$ matrix with $x$ as first row and $y$ as second row. This introduces a sign ambiguity in the definition of $\psi_i^l$ due to the order of $j$ and $k$. There is a "good" way to make such a choice; we will suppress this for now, but for details we refer the reader to Lemma 11.6.1.

**Proposition 6.3.1.** *With the notations above, we have*

$$\begin{aligned} \{\Pi\}\sqrt{\gamma\Big(\frac{1}{2}, S^-\Big)} &= \nu_{\mathbb{P}}^{-2}\, \mathbf{I}^{\mathbf{E}}(\mathbb{P}^1_F, \psi) \\ &= \nu_{\mathbb{P}}^{-2} \int_{\mathcal{R}\backslash(\mathbb{P}^1)^{\mathbf{E}}} \prod_{il \in \mathbf{O}} \left( \frac{\chi_{ij}\chi_{ik}}{\chi_{il}} |\cdot|^{-\frac{1}{2}} \right) (y_{ij} \wedge y_{ik}), \end{aligned} \tag{6.3.1}$$

*where $\nu_{\mathbb{P}}$ is as in* (3.2.3). *The integrals are* absolutely convergent *for $\chi$ unitary. (Note that the second expression is simply the explication of the notation; in it,* $jk \in \mathbf{E}$ *is the opposite edge to that defined by* $il$.*)*

An outline of the proof towards this formula will be sketched shortly in § 6.4, and the details will be contained in § 11.

Although (6.3.1) is compact, it is a little opaque, so we offer two different reinterpretations of it. The first will be given now, and the second in § 7. This first interpretation will also help us see that the formula is absolutely convergent when $\psi$ is unitary.

**6.3.2.** *Interpretation of* (6.3.1) *in terms of moduli of six points on the projective line.* Let us consider the space $\mathbf{M}^\circ$ of configurations of six points on $\mathbb{P}^1$ indexed by $\mathbf{E}$ where the points indexed by incident edges are distinct (thus, the points indexed by opposite edges may collide). $\mathrm{PGL}_2$ acts freely on this space, and upon taking the quotient we arrive at a moduli space

$$\mathcal{M}^\circ = \mathbf{M}^\circ/\mathrm{PGL}_2,$$

which more concretely amounts to configurations of three points with certain distinctness properties, see (6.3.3) below.

We first define a section $\Omega$ of the square of the canonical sheaf $\omega^2$ on $\mathcal{M}^\circ$. First of all, consider the formula

$$\Omega_{\mathbf{M}} = \frac{\left(\bigwedge_e dx_e\right)^2}{\prod_{e\neq e'}(x_e - x_{e'})}, \tag{6.3.2}$$

where $e, e'$ range over pairs of edges sharing a vertex; the overall sign depends on the ordering in the numerator, but this will not matter for us. It defines a section of the square $\omega^2$ of the canonical bundle, on the open subset of $\mathbf{M}^\circ$ where the $x_e$ actually lie inside $\mathbb{A}^1 \subset \mathbb{P}^1$. This section, however, extends to $\mathbf{M}^\circ$. The form $\Omega_{\mathbf{M}}$ is $\mathrm{PGL}_2$-invariant, and we can "divide it" by the square of the fixed algebraic volume form on $\mathrm{PGL}_2$ (see (3.2.2)) to arrive at a corresponding section $\Omega$ of the square of the canonical sheaf on $\mathcal{M} = \mathbf{M}/\mathrm{PGL}_2$ itself.

*Remark* 6.3.3. For later use we explicate both $\mathcal{M}^\circ$ and $\Omega$. We may identify

$$\mathcal{M}^\circ \simeq \{(x, y, z) \in \mathbb{P}^1 : x \neq y, y \neq z, x \notin \{1, \infty\}, y \notin \{\infty, 0\}, z \notin \{0, 1\}\} \tag{6.3.3}$$

To do so, we use the $\mathrm{PGL}_2$-action to identify $\mathcal{M}^\circ$ with the subset of $\mathbf{M}^\circ$ where $x_{\mathbf{14}} = 0, x_{\mathbf{13}} = 1, x_{\mathbf{23}} = \infty$, and take $x = x_{\mathbf{12}}, y = x_{\mathbf{24}}, z = x_{\mathbf{34}}$ for the remaining coordinates. Upon comparing (6.3.2) with (3.2.2), we deduce that the form $\Omega$ is given by

$$\Omega = \frac{(dx \wedge dy \wedge dz)^2}{xyz(x-y)(y-z)(1-z)(1-x)}. \tag{6.3.4}$$

Next, we define a morphism

$$\mu \colon \mathcal{M}^\circ \to \tilde{\mathsf{D}}_6 \otimes \mathbb{G}_m \simeq \mathbb{G}_m^6,$$

by requiring that, for $ij \in \mathbf{O}$, the $ij$ coordinate of $\mu$ is given by the following rule: move $x_{ij}$ to $\infty$ by means of a projective transformation, and then set

$$\mu_{ij} = \frac{x_a - x_b}{x_c - x_d}, \tag{6.3.5}$$

where the three distinct edges $a, b, e$ meet at $i$ and the three distinct edges $c, d, e$ meet at $j$. Again, there are choices of ordering here; we make them so that $\mu_{ij}\mu_{ji} = 1$. This is not unique, but different choices only affect the $\mu_{ij}$s by a sign.

Now $|\Omega|^{\frac{1}{2}}$ makes sense as a a volume form on $\mathcal{M}^\circ(F)$ for $F$ a local field, so long as we fix a Haar meausre on $F$.[12] This permits us to state the following:

[12] Indeed, for each point $x_0 \in \mathcal{M}^\circ(F)$ we choose an analytic isomorphism $\Phi : U \simeq U_{x_0}$ from an open subset $U \subset F^n$ onto an analytic neighbourhood $U_{x_0}$ of $x_0$ in $\mathcal{M}^\circ(F)$; analytic means that it is given by convergent $F$-valued power series. The pullback of $\Omega$ by means of this form has the shape $A(z_1, \ldots, z_n)(dz_1 \wedge \cdots \wedge dz_n)^2$

**Proposition 6.3.4.** *We have an equality, up to sign,*

$$I^{\mathbf{E}}(\mathbb{P}^1_F, \psi) = \nu_{\mathbb{P}} \int_{\mathcal{M}^\circ(F)} \prod_{i<j} \chi_{ij}(\mu_{ij}(x)) \cdot |\Omega|^{\frac{1}{2}} \tag{6.3.6}$$

*where* $\nu_{\mathbb{P}}$ *is as in* (3.2.3), $\mu$ *is as in* (6.3.5) *and* $\mu_{ij}$ *is the coordinate of* $\mu$ *corresponding to* $ij \in \mathbf{E}$.

*Proof.* This is simply a rewriting of the second line of (6.3.1). In carrying out this verification, it is convenient to use a more intrinsic formulation of (6.3.5): if $x_{ij}$ are homogeneous coordinates of $\mathbb{P}^1$ corresponding to $ij$, then

$$\mu_{ij} = \frac{(x_{ij} \wedge y_i)(x_{ij} \wedge z_i)(y_j \wedge z_j)}{(x_{ij} \wedge y_j)(x_{ij} \wedge z_j)(y_i \wedge z_i)}$$

where $y_i, z_i$ (resp. $y_j, z_j$) correspond to the edges other than $ij$ connected to $i \in \mathbf{V}$ (resp. $j \in \mathbf{V}$); we recover our prior description by moving $x_{ij}$ to $\infty$. Yet again, the sign of these wedges above is of no concern to us. Finally, the factor $\nu_{\mathbb{P}}$ comes from the fact that in $I^{\mathbf{E}}(\mathbb{P}^1_F, \psi)$ we used the Haar measure on $\mathcal{R}$ rather than the measure induced by $(\mathbb{P}^1)^3$ to define the measure on $\mathcal{R}\backslash(\mathbb{P}^1)^{\mathbf{E}}$. ∎

With this intepretation at hand, we will prove the convergence claim of Proposition 6.3.1 in § 10.2.

**6.4. Outline of the argument for** (6.3.1)**.** The proof requires us to be careful about isomorphisms between different models for the same representation, in a way that will seem excessively pedantic. However this is the price we pay for having a very short definition of the symbol: complexity gets transferred to the set-up phase of computations.

For each $ij \in \mathbf{O}$, let $\pi_{ij}$ be the induced representation from $\chi_{ij}$; it is realized in the space of sections of a line bundle on $\mathbb{P}^1$, as described in detail in § 11.2. Note that $\pi_{ij} \neq \pi_{ji}$, but they are isomorphic; in fact, by definition of the principal series model, the fact that $\chi_{ij}\chi_{ji} = 1$ will give rise to a natural pairing

$$\pi_{ij} \times \pi_{ji} \longrightarrow \mathbb{C}.$$

Fix a splitting $\mathbf{E} \to \mathbf{O}$, that is to say, a choice of orientation of each edge. For each such chosen orientation $ij \in \mathbf{O}$, we fix a symmetric pairing on $\pi_{ij}$ and an isomorphism $I_e \colon \pi_{ij} \to \pi_{ji}$, and we transport the symmetric pairing from $\pi_{ij}$ to $\pi_{ji}$ by means of the fixed isomorphism $I_e$.

Put

$$\Pi_G = \bigotimes_{ij \in \mathbf{O}} \pi_{ij} = \bigotimes_{ij \in \mathbf{E}} (\pi_{ij} \otimes \pi_{ji}),$$
$$\Pi'_G = \bigotimes_{ij \in \mathbf{E}} (\pi_{ij} \otimes \pi_{ij}),$$

---

for an analytic function $A$. The integral of $f$ over the analytic neighbourhood $U_{x_0}$ is then declared to be the integral of $(f \circ \Phi) \times |A|^{\frac{1}{2}}$ against Haar measure on $U$.

where we use the splitting to choose an ordering in the second definition. Both $\Pi_G, \Pi'_G$ are endowed with symmetric self-pairings arising from those on $\pi_{ij}$. The representations $\Pi'_G$ and $\Pi_G$, together with these pairings, are isomorphic by means of the chosen isomorphisms $I_e$:

$$I = \otimes I_e \colon \big(\Pi'_G, (-,-)\big) \xrightarrow{\sim} \big(\Pi_G, (-,-)\big).$$

Our definition of the tetrahedral symbol is expressed in terms of $\Pi'_G$. But unfortunately the contraction functional does not look nice in the natural induced model for $\Pi'_G$. This problem is remedied by switching to $\Pi_G$ instead: it is realized as the space of sections of a line bundle on $(\mathbb{P}^1)^{\mathbf{O}}$; and a D-invariant functional $\phi^D$ on $\Pi_G$ given simply by integration over the "diagonal" $(\mathbb{P}^1)^{\mathbf{E}}$, i.e. the *closed* D-orbit, whereas an H-invariant functional $\phi^H$ on $\Pi_G$ is given by integration over the *open* H-orbit on $(\mathbb{P}^1)^{\mathbf{O}}$. After computing the constant of proportionality relating $I^*\phi^H$ and $\Lambda^H$, as well as the constant of proportionality relating $I^*\phi^D$ and $\Lambda^D$ we are reducing to answering the question: when we average $\phi^D$ over H, which multiple of $\phi^H$ do we get?

When we average $\phi^D$ over H, we in effect are pushing forward by means of the map

$$H \times (\mathbb{P}^1_F)^{\mathbf{E}} \longrightarrow (\mathbb{P}^1_F)^{\mathbf{O}}.$$

From this analysis, it follows that $\{\Pi\}$ is an integral over the fibers of this map. Now, the projection map from a general fiber onto $(\mathbb{P}^1_F)^{\mathbf{E}}$ is readily verified to be a birational morphism. This gives us an expression for $\{\Pi\}$ as an integral over $(\mathbb{P}^1_F)^{\mathbf{E}}$. After we work out the details in § 11 we get (6.3.1).

## 7. Hypergeometric formulas for the tetrahedral symbol

In this section, $\mathcal{R}$ is split and $\pi_{ij}$ are irreducible principal series. We will describe formulas for the tetrahedral symbol of hypergeometric type (Theorem 7.2.1), and in the case $F = \mathbb{R}$ we will simplify the resulting formula to a sum of ${}_4F_3$ hypergeometric series evaluated at the special point $z = 1$.

**7.1. Hypergeometric functions on Grassmannians.** Now let $k < n$ be integers. Let $\underline{\chi} = (\chi_1, \dots, \chi_n)$ be $n$ characters of $F^\times$ with the property that

$$\prod_i \chi_i = |\cdot|^{-k}.$$

Let $X \subset F^n$ be a k-dimensional subspace. Then the integral

$$\int_{\mathbb{P}X} \underline{\chi} := \int_{\mathbb{P}X} \prod_i \chi_i(x_i). \tag{7.1.1}$$

makes sense at least formally, as the function is $-k$ homogeneous and can be integrated over the projective space $\mathbb{P}X$, according to the general procedure of § 2.4. Such functions for $F = \mathbb{R}$ were studied by Aomoto, Gelfand and others. They can be considered as generalizations of Gauss's hypergeometric function, which corresponds to the case when $k = 2, n = 4$, and $X$ is spanned by the rows of the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & t \end{bmatrix}$$

where $t \in \mathbb{R}$ is a fixed parameter.

**7.2. The hypergeometric formula for the tetrahedral symbol.** Now let $X$ be the subspace of $F^8$ (so that $k = 4, n = 8$) of the form

$$X = (x-y, y-z, z-w, w-x, x, y, z, w) \subset F^8,$$

and define an 8-tuple of quasicharacters

$$\underline{\chi} = \left(\chi^3_{2-}, \chi^1_{4-}, \chi^2_{3-}, \chi^4_{1-}, \chi^3_{1-}, \chi^3_{4-}, \chi^2_{4-}, \chi^2_{1-}\right),$$

where we use notation similar to that of § 6.3:

$$\chi^l_i := \frac{\chi_{ij}\chi_{ik}}{\chi_{il}}.$$

and recall the subscript "$-$" means that we multiply by $|\cdot|^{-\frac{1}{2}}$.

**Theorem 7.2.1.** *Let $\chi \in \mathcal{X}_0$, with the notations of* (5.2.1)*, and let $X$ and $\underline{\chi}$ be as above. Then the tetrahedral symbol associated to $\chi$ can be expressed as a hypergeometric integral, where we integrate $\underline{\chi}$ over the projectivization of $X$:*

$$\begin{Bmatrix} \chi_{12} & \chi_{13} & \chi_{14} \\ \chi_{34} & \chi_{24} & \chi_{23} \end{Bmatrix} \cdot \sqrt{\gamma\Big(\frac{1}{2}, S^-\Big)} = \nu_{\mathbb{P}}^{-1} \int_{\mathbb{P}X} \underline{\chi}, \tag{7.2.1}$$

*and both sides are absolutely convergent for $\chi \in \mathcal{X}_0$.*

*Proof of Theorem 7.2.1 from Proposition 6.3.4.* The right-hand side of (7.2.1), aside from the factor $\nu_{\mathbb{P}}^{-1}$, is the following integral:

$$\int_{[x,y,z,w]\in\mathbb{P}^3(F)} \chi^3_{2-}(x-y)\chi^1_{4-}(y-z)\chi^2_{3-}(z-w)\chi^4_{1-}(w-x)\chi^3_{1-}(x)\chi^3_{4-}(y)\chi^2_{4-}(z)\chi^2_{1-}(w).$$

which, more explicitly, turns into the following, upon de-homogenization and expansion:

$$\begin{aligned} &\int_{F^3} \chi^3_{2-}(x-y)\chi^1_{4-}(y-z)\chi^2_{3-}(z-1)\chi^4_{1-}(1-x)\chi^3_{1-}(x)\chi^3_{4-}(y)\chi^2_{4-}(z)dxdydz \\ =&\int_{F^3} \chi_{12}\left(\frac{x(1-x)}{x-y}\right)\chi_{13}\left(\frac{1-x}{x(z-1)}\right)\chi_{14}\left(\frac{x(y-z)}{(1-x)yz}\right)\chi_{23}\left(\frac{z-1}{x-y}\right)\chi_{24}\left(\frac{(x-y)z}{y(y-z)}\right) \\ &\qquad \chi_{34}\left(\frac{y(z-1)}{z(y-z)}\right)|xyz(x-y)(y-z)(z-1)(1-x)|^{-\frac{1}{2}}dxdydz. \end{aligned} \tag{7.2.2}$$

This amounts to the explicit content of Proposition 6.3.4, once we use the explicit coordinates provided by Remark 6.3.3, drawn as below:

$$\begin{tikzcd}
 & & \mathbf{1} \arrow[dll, "x"'] \arrow[d, "0"] & & \\
 & & \mathbf{4} & & \\
\mathbf{2} \arrow[urr, "y"] \arrow[rrrr, "\infty"] & & & & \mathbf{3} \arrow[ull, "z"'] \arrow[uull, "w"']
\end{tikzcd} \tag{7.2.3}$$

Examining the labelling from this remark, we readily verify that the arguments of the $\chi_{ij}$ are just the $\mu_{ij}$ from Proposition 6.3.4. ∎

**7.3. The tetrahedral symbol as a convolution of $\gamma$-functions.** The following striking formula can be derived, after some manipulation, from Theorem 7.2.1, and uses the same notation (we also remind the reader of the notations in § 2.2.5):

**Proposition 7.3.1.** *Abridge $\rho_{ij} = \{\chi_{ij}, \chi_{ji}\}$, Define four-element subsets $A, B$ of characters as, respectively, the union of $\chi_{\mathbf{41}} \otimes \rho_{\mathbf{12}}$ and $\chi_{\mathbf{23}} \otimes \rho_{\mathbf{43}}$, and the union of $\rho_{\mathbf{13}}$ and $\chi_{\mathbf{23}}\chi_{\mathbf{41}} \otimes \rho_{\mathbf{24}}$. Then*

$$\begin{Bmatrix} \chi_{\mathbf{12}} & \chi_{\mathbf{13}} & \chi_{\mathbf{14}} \\ \chi_{\mathbf{34}} & \chi_{\mathbf{24}} & \chi_{\mathbf{23}} \end{Bmatrix} = \frac{\nu_{\mathbb{P}}^{-1} \int_{\mu} \gamma(\frac{1}{2} + \epsilon, A \otimes \mu)\gamma(1-\epsilon, B^{-1} \otimes \mu^{-1}) d\mu}{\sqrt{\gamma(\frac{1}{2}, A \otimes B^{-1})}}, \tag{7.3.1}$$

*where $\mu$ ranges over characters of $F^{\times}$, whereas $\epsilon$ is any complex number with real part between $0$ and $\frac{1}{2}$.*

The proof will be given in § 14.5.

**7.4. The case $F = \mathbb{R}$: expression in terms of ${}_4F_3$.** Using Mellin transform, we can relate the tetrahedral symbol to generalized hypergeometric series. For simplicity, we assume

$$\chi_{\mathbf{12}} = |\cdot|^{J_1}, \quad \chi_{\mathbf{31}} = |\cdot|^{J_2}, \quad \chi_{\mathbf{14}} = |\cdot|^{J_3},$$
$$\chi_{\mathbf{34}} = |\cdot|^{J_4}, \quad \chi_{\mathbf{24}} = |\cdot|^{J_5}, \quad \chi_{\mathbf{23}} = |\cdot|^{J_6},$$

where $J_1, \ldots, J_6 \in \mathbb{C}$. Here, we choose $\chi_{\mathbf{31}}$ instead of $\chi_{\mathbf{13}}$ only because in (4.3.1) the blackbold $\mathbb{2}$ is the label of **31**; the orientations themselves have no impact here. The general case can be derived with the same method, but we only give explicit statement about this special case. We use shorthands such as $J_{\bar{1}23} = -J_1 + J_2 + J_3$, $J_{235\bar{6}} = J_2 + J_3 + J_5 - J_6$, and so on.

Then in § 14.6 we will show that we can express the tetrahedral symbol as a sum of ${}_4F_3$s in two different ways:

$$\{\Pi\}\sqrt{L\left(\frac{1}{2}, S\right)} = -\frac{L(\frac{1}{2}, S_1^*)}{\gamma(J_{\bar{2}\bar{3}\bar{5}6})\gamma(J_{\bar{2}\bar{3}56})\gamma(J_{\bar{2}\bar{2}})} {}_4F_3\left( \begin{matrix} J_{\bar{1}23} + \frac{1}{2}, J_{123} + \frac{1}{2}, J_{24\bar{6}} + \frac{1}{2}, J_{2\bar{4}\bar{6}} + \frac{1}{2} \\ J_{235\bar{6}} + 1, J_{23\bar{5}\bar{6}} + 1, J_{22} + 1 \end{matrix} \,\middle|\, 1 \right)$$

$$\begin{aligned}
&-\frac{L(\frac{1}{2},S_2^*)}{\gamma(J_{235\bar{6}})\gamma(J_{55})\gamma(J_{\bar{2}35\bar{6}})}{}_4F_3\left(\begin{matrix} J_{\bar{1}\bar{5}6}+\frac{1}{2}, J_{1\bar{5}6}+\frac{1}{2}, J_{\bar{3}4\bar{5}}+\frac{1}{2}, J_{\bar{3}\bar{4}\bar{5}}+\frac{1}{2} \\ J_{\bar{2}\bar{3}\bar{5}6}+1, J_{\bar{5}\bar{5}}+1, J_{2\bar{3}\bar{5}6}+1 \end{matrix}\middle| 1\right)\\
&-\frac{L(\frac{1}{2},S_3^*)}{\gamma(J_{23\bar{5}\bar{6}})\gamma(J_{\bar{5}\bar{5}})\gamma(J_{\bar{2}3\bar{5}\bar{6}})}{}_4F_3\left(\begin{matrix} J_{\bar{1}56}+\frac{1}{2}, J_{156}+\frac{1}{2}, J_{\bar{3}45}+\frac{1}{2}, J_{\bar{3}\bar{4}5}+\frac{1}{2} \\ J_{\bar{2}\bar{3}56}+1, J_{55}+1, J_{2\bar{3}56}+1 \end{matrix}\middle| 1\right)\\
&-\frac{L(\frac{1}{2},S_4^*)}{\gamma(J_{22})\gamma(J_{2\bar{3}\bar{5}6})\gamma(J_{2\bar{3}56})}{}_4F_3\left(\begin{matrix} J_{\bar{1}\bar{2}3}+\frac{1}{2}, J_{1\bar{2}3}+\frac{1}{2}, J_{\bar{2}4\bar{6}}+\frac{1}{2}, J_{\bar{2}\bar{4}\bar{6}}+\frac{1}{2} \\ J_{\bar{2}\bar{2}}+1, J_{\bar{2}35\bar{6}}+1, J_{\bar{2}3\bar{5}\bar{6}}+1 \end{matrix}\middle| 1\right),
\end{aligned}$$

and

$$\begin{aligned}
\{\Pi\}\sqrt{L\Big(\frac{1}{2},S\Big)} = &+\frac{L(\frac{1}{2},S_{1'}^*)}{\gamma(J_{11})\gamma(J_{1\bar{3}46})\gamma(J_{1\bar{3}\bar{4}6})}{}_4F_3\left(\begin{matrix} J_{\bar{1}23}+\frac{1}{2}, J_{\bar{1}\bar{5}6}+\frac{1}{2}, J_{\bar{1}56}+\frac{1}{2}, J_{\bar{1}\bar{2}3}+\frac{1}{2} \\ J_{\bar{1}\bar{1}}+1, J_{\bar{1}3\bar{4}\bar{6}}+1, J_{\bar{1}34\bar{6}}+1 \end{matrix}\middle| 1\right)\\
&+\frac{L(\frac{1}{2},S_{2'}^*)}{\gamma(J_{\bar{1}\bar{1}}\gamma(J_{\bar{1}\bar{3}46})\gamma(J_{\bar{1}\bar{3}\bar{4}6})}{}_4F_3\left(\begin{matrix} J_{123}+\frac{1}{2}, J_{1\bar{5}6}+\frac{1}{2}, J_{156}+\frac{1}{2}, J_{1\bar{2}3}+\frac{1}{2} \\ J_{11}+1, J_{13\bar{4}\bar{6}}+1, J_{134\bar{6}}+1 \end{matrix}\middle| 1\right)\\
&+\frac{L(\frac{1}{2},S_{3'}^*)}{\gamma(J_{\bar{1}3\bar{4}6})\gamma(J_{13\bar{4}6})\gamma(J_{\bar{4}\bar{4}})}{}_4F_3\left(\begin{matrix} J_{24\bar{6}}+\frac{1}{2}, J_{\bar{3}4\bar{5}}+\frac{1}{2}, J_{\bar{3}45}+\frac{1}{2}, J_{\bar{2}4\bar{6}}+\frac{1}{2} \\ J_{1\bar{3}4\bar{6}}+1, J_{\bar{1}\bar{3}4\bar{6}}+1, J_{44}+1 \end{matrix}\middle| 1\right)\\
&+\frac{L(\frac{1}{2},S_{4'}^*)}{\gamma(J_{\bar{1}346})\gamma(J_{1346})\gamma(J_{44})}{}_4F_3\left(\begin{matrix} J_{2\bar{4}\bar{6}}+\frac{1}{2}, J_{\bar{3}\bar{4}\bar{5}}+\frac{1}{2}, J_{\bar{3}\bar{4}5}+\frac{1}{2}, J_{\bar{2}\bar{4}\bar{6}}+\frac{1}{2} \\ J_{1\bar{3}\bar{4}\bar{6}}+1, J_{\bar{1}\bar{3}\bar{4}\bar{6}}+1, J_{\bar{4}\bar{4}}+1 \end{matrix}\middle| 1\right).
\end{aligned}$$

Here $S_1^*,\dots,S_4^*$ and $S_{1'}^*,\dots,S_{4'}^*$ are certain slightly modified versions of $S^-$ (details in § 14.6). For generic $J_1,\dots,J_6$, the eight ${}_4F_3$s on both right-hand sides turn out to be absolutely convergent at 1 and so the expressions are well-defined. See, also, § 9.4 for a discussion of related results in the literature.

In particular, our results imply that *there exists a* 4*-term combination of* ${}_4F_3$*s whose value at* 1 *satisfies* $W(D_6)$*-symmetry.* We have not been able to find this statement in the literature; it fits in a broader patterm of extra symmetry enjoyed by evaluations of generalized hypergeometric series, cf. [FGS11]. In this connection, we observe that the Thomae symmetry of ${}_3F_2$ hypergeometric series is, in the same fashion, related to the Regge symmetries of 3j symbols.

## 8. Langlands duality; the tetrahedral symbol as an interface

We now sketch a picture that is Langlands dual to the tetrahedral symbol. It does not shed much light on the proofs, and it may seem as if we are simply translating into an arcane tongue. But it was instrumental to the discovery of our results; it brings out the role of the *group* $\mathrm{Spin}_{12}$ rather than merely its maximal torus and Weyl group; and, although we do not pursue it here, this language suggests natural avenues of generalization.

**8.1. Tetrahedral symbol via a correspondence.** First of all, let us explain how the definition of the tetrahedral symbol is a special case of an invariant attached to a correspondence between spherical varieties. Recall the notation of (3.2.1):

$$G = \mathcal{R}^{\mathbf{O}}, \quad D = \mathcal{R}^{\mathbf{E}}, \quad H = \mathcal{R}^{\mathbf{V}},$$

so that $G \simeq \mathcal{R}^{12}, D \simeq \mathcal{R}^6, H \simeq \mathcal{R}^4$. Let $X, Y$ be the corresponding homogeneous spaces:

$$X = D\backslash G, \quad Y = H\backslash G.$$

Let $C^\infty(X)$ (resp. $C^\infty(Y)$) be the space of smooth $\mathbb{C}$-valued functions on X (resp. Y). We also regard elements in $C^\infty(X)$ as functions on G that are D-invariant on the left, and similarly for $C^\infty(Y)$. With $\Pi_G$ as defined in § 3.3, we consider the diagram of G-representations

$$\begin{tikzcd} & C^\infty(X) \arrow[dd, dashed, "\mathrm{Av}"] \\ \Pi_G \arrow[ur, "\Lambda^X"] \arrow[dr, "\Lambda^Y"'] & \\ & C^\infty(Y) \end{tikzcd} \tag{8.1.1}$$

where the morphisms of $\Pi_G$ into $C^\infty(X)$ and $C^\infty(Y)$ are "distinguished" embeddings, which is to say that their values at the identity coset are given by the normalized functionals $\Lambda^D, \Lambda^H$ that were specified earlier; the vertical map is the averaging integral:

$$\mathrm{Av}(f)(x) = \int_{(D\cap H)\backslash H} f(hx)dh.$$

The Haar measure is fixed as in the beginning of the paper, and we *assume* that Av converges absolutely on the image of $\Pi_G$ (which turns out to be true for tempered $\Pi$). We can reinterpret the tetrahedral symbol $\{\Pi\}$ as *describing the scalar by which this diagram fails to commute*.

**8.2. Interface and its dual.** The key ingredients in the above definition are two spaces $X, Y$ and the correspondence

$$X \longleftarrow Z \longrightarrow Y$$

where $Z := (D \cap H)\backslash G$; the averaging operator amounts to pulling-back and pushing-forward along this diagram. Now an important theme of relative Langlands duality is that, in order to achieve symmetry between automorphic and Galois sides, it is convenient to switch to symplectic geometry, i.e. to re-code spaces by their cotangent bundles. From this point of view we should replace X and Y by $M = T^*X, N = T^*Y$ and replace Z by the *Lagrangian correspondence*

$$L := \text{conormal to } Z \subset M \times N.$$

In the language of [BZSV24], this defines an *interface* between the hyperspherical varieties M and N, and it is reasonable to search for a *dual interface*

$$\check{L} \subset \check{M} \times \check{N},$$

where $\check{M}, \check{N}$ are the respective relative Langlands duals for $M, N$. The paper [BZV] does this for various naturally arising classes of L; however, in all examples of that paper, both L and $\check{L}$ are smooth. The tetrahedral symbol offers one of the first interesting cases where this is not so.

The dual group of G is

$$\check{G} = SL_2(\mathbb{C})^{\mathbf{O}} \simeq SL_2(\mathbb{C})^{12}.$$

Consider also the dual $\check{D} = SL_2^{\mathbf{E}} \simeq SL_2(\mathbb{C})^6$ which we regard as a subgroup of $\check{G}$ in the obvious way, i.e., corresponding to the map $\mathbf{O} \to \mathbf{E}$ where we forget orientation. (Note that, in general, an embedding $D \subset G$ does not induce an embedding of dual groups; but in the current case we just use the obvious embedding.)

According to relative Langlands duality, the Hamiltonian varieties $M = T^*X$ and $N = T^*Y$ are respectively dual to the $\check{G}$-spaces

$$\check{M} = T^*(\check{G}/\check{D}), \quad \check{N} \simeq \bigoplus_{i \in \mathbf{V}} \mathbb{C}^2_{ij} \otimes \mathbb{C}^2_{ik} \otimes \mathbb{C}^2_{il}.$$

In fact, $\check{N}$ is the pull-back of a half-spin representation $S$ of $\mathrm{Spin}_{12}$ by means of the homomorphism

$$\tau \colon \check{D} = SL_2^{\mathbf{E}} \longrightarrow \mathrm{Spin}_{12}$$

that was described in (4.2.1). Note that are *two* half-spin representation of $\mathrm{Spin}_{12}$, only one of which restricts to $\check{N}$ (see § 4.4 for more details). Accordingly it makes sense to consider within $\check{N}$ the cone $P \subset S \simeq \check{N}$ of pure spinors. The reader can refer to § 13.2 for a description of $P$ as a subvariety of $S$.

We are now ready to describe what we think is the picture in relative Langlands duality that underlies the theory of the tetrahedral symbol.

> *Key proposal:* The dual interface to the tetrahedral symbol is the induction of the cone of pure spinors $P \subset \check{N}$ from $\check{D}$ to $\check{G}$:
>
> $$\check{L} = \check{G} \times^{\check{D}} P. \tag{8.2.1}$$
>
> where the projection from $\check{L}$ to $\check{M}$ factors through the zero section $\check{G}/\check{D} \to \check{M}$, and the projection to $\check{N}$ is given by $(g, v) \mapsto gv$.

This picture, for example, "explains" Theorem 5.2.1, in a sense discussed further in § 9.6 and § 9.7.

## 9. Further topics

What we have proved in this paper gives, we think, a deeper context for the Regge symmetry of the classical 6j symbols. However, the classical 6j symbols has many other beautiful properties too, and it would be interesting to study these from the point of view taken in this paper. We will give a brief discussion of a few such points here.

The classical 6j symbol plays a distinguished role in the theory of orthogonal polynomials: it gives the most general class of orthogonal polynomials in the *Askey scheme* of orthogonal-hypergeometric polynomials. See [Lab85, Koo88]. Classical orthogonal polynomials satisfy both orthogonality relations and difference equations; and in § 9.1 and § 9.2 we discuss, respectively, the orthogonality and difference equations that hold in the context of this paper.

Secondly, the tetrahedral symbol has played an important role in analytic number theory (although apparently this connection has not been explicitly made previously). In the first-named author's work on the subconvexity problem for L-functions, a key role was played by a certain spectral reciprocity formula relating two sums of L-functions. In this formula, as observed in [MV10], there is a somewhat mysterious integral transform. This is, as we shall sketch in § 9.3, precisely the integral transform associated by the two-variable function obtained by fixing four of the six inputs to the tetrahedral symbol.

The relationship of the tetrahedral symbol to relative Langlands duality could be more deeply understood in many ways. In § 9.5 we use the general formulation of Langlands duality to discuss the situation beyond the principal series case. In § 9.6 we formulate a conjecture relating the tetrahedral symbol to the representation theory of $\mathrm{Spin}_{12}$, and in § 9.7 we discuss associated questions in geometric representation theory.

We do not touch on another important role of the 6j symbol: namely, its role in the Turaev–Viro [TV92] invariant of 3-manifolds; it would be interesting to look at this, too, from the dual viewpoint.

As this section is primarily meant to serve as a source of motivation for futher work, we will not always give full details, particularly around issues of convergence.

**9.1. Orthogonality.** Like many classical special functions, the tetrahedral symbol has remarkably rigid orthogonality properties.

Fix a pair of opposite edges of the tetrahedron: let us take them to be $\{\mathbf{13}, \mathbf{24}\}$ and assign tempered representations to the remaining 4 edges. We can then regard the tetrahedral symbol as a function of $\sigma = \pi_{\mathbf{13}}, \tau = \pi_{\mathbf{24}}$, which we denote by $\{\Pi\}_{\sigma\tau}$, and which we will regard as a function

$$\begin{aligned} \mathcal{P}_0 \times \mathcal{P}_0 &\longrightarrow \mathbb{C}, \\ (\sigma, \tau) &\longmapsto \{\Pi\}_{\sigma\tau}, \end{aligned}$$

where we recall that $\mathcal{P}_0$ is the *tempered dual* of $\mathcal{R}$, i.e. the set of irreducible tempered representations up to isomorphism.

The support of this function is a direct product, i.e. if $\{\Pi\}_{\sigma_0\tau_0}$ and $\{\Pi\}_{\sigma_1\tau_1}$ are nonzero, then $\{\Pi\}$ is not identically zero on the component of $\mathcal{P}_0 \times \mathcal{P}_0$ containing $(\sigma_0, \tau_1)$. Indeed, the condition for $\{\Pi\}$ to be not identically vanishing on the component containing $(\sigma, \tau)$ is that all four of the $\mathcal{R}^3$-representations

$$\pi_{\mathbf{12}} \otimes \sigma \otimes \pi_{\mathbf{14}}, \quad \pi_{\mathbf{32}} \otimes \sigma \otimes \pi_{\mathbf{34}}, \quad \pi_{\mathbf{21}} \otimes \tau \otimes \pi_{\mathbf{23}}, \quad \pi_{\mathbf{41}} \otimes \tau \otimes \pi_{\mathbf{43}}$$

admit $\mathcal{R}$-invariant functionals.

Now this $\mathcal{P}_0$ has a natural Borel structure; and a choice of Haar measure on $\mathcal{R}$ determines upon $\mathcal{P}_0$ a canonical measure, the Plancherel measure, characterized by the fact that for a continuous compactly supported function $f$ on $\mathcal{R}$ we have

$$f(e) = \int_{\tau \in \mathcal{P}_0} \mathrm{Tr}_\tau(f) d\tau. \tag{9.1.1}$$

Then the orthogonality is expressed by means of:

**Proposition 9.1.1.** *The function $\{\Pi\}_{\sigma\tau}$ is the kernel of a unitary transformation:*

$$L^2(\mathcal{P}_0^{(a)}) \longrightarrow L^2(\mathcal{P}_0^{(b)}),$$
$$f \longmapsto \left(\sigma \mapsto \int \{\Pi\}_{\sigma\tau} f(\tau) d\tau\right).$$

*and where $\mathcal{P}_0^{(a)} \times \mathcal{P}_0^{(b)}$ is the union of all components of $\mathcal{P}_0 \times \mathcal{P}_0$ on which $\{\Pi\}_{\sigma\tau}$ is not identically zero.*

We sketch a proof in § 15.1. Actually, we will establish a more precise statement, which we now explain. Set

$$\Sigma := \pi_{12} \otimes \pi_{23} \otimes \pi_{34} \otimes \pi_{41},$$

an irreducible representation of $\mathcal{R}^4$. A choice of $\sigma = \pi_{13}$ and $\tau = \pi_{24}$ determines $\mathcal{R}$-invariant functionals

$$E_{13} = E_\sigma \in \Sigma^* \text{ and } E_{24} = E_\tau \in \Sigma^* \tag{9.1.2}$$

in the following way. First fix symmetric self-duality pairings on all $\pi_{ij}$ including $\sigma, \tau$. Consider first

$$\Sigma \otimes \sigma \otimes \sigma \simeq \underbrace{(\pi_{32} \otimes \pi_{34} \otimes \pi_{13})}_{\pi_{O_3}} \otimes \underbrace{(\pi_{13} \otimes \pi_{14} \otimes \pi_{12})}_{\pi_{O_1}}$$

We normalize functionals $\Lambda_3$ on the first factor $\pi_{O_3}$ and $\Lambda_1$ on the second factor $\pi_{O_1}$ by means of the same recipe as in (3.4.3). To produce $E_\sigma$, contract in the $\sigma$ variables to produce a class in $\Sigma^*$, that is to say, sum over $e_i \otimes e_i \in \sigma \otimes \sigma$ where $e_i$ is an orthonormal basis with respect to the fixed symmetric self-pairing on $\sigma$; it is presumably not difficult to verify the convergence of this summation. The construction of $E_\tau$ is precisely parallel using instead

$$\Sigma \otimes \tau \otimes \tau \simeq \pi_{O_2} \otimes \pi_{O_4}$$

and forming functionals $\Lambda_2, \Lambda_4$ on the two factors.

**Proposition 9.1.2.** *There is a natural Hilbert structure on $(\Sigma^*)^{\mathcal{R}}$ with the property that the rules*

$$f \mapsto \int f(\sigma) E_\sigma d\sigma, \quad g \mapsto \int g(\tau) E_\tau d\tau$$

*extend to isometries*

$$L^2(\mathcal{P}_0^{(a)}) \text{ or } L^2(\mathcal{P}_0^{(b)}) \longrightarrow \text{Hilbert completion of } \Sigma^*.$$

*Moreover, the two isometries are intertwined by the transformation of Proposition 9.1.1.*

**9.2. Difference and differential equations.** Specialize to the case $F = \mathbb{R}$; a variant of the following discussion applies for $F = \mathbb{C}$ too. In these cases, like many classical special functions, the tetrahedral symbol then satisfies a large system of difference equations:

Quasi-characters of $F^\times$ all have the form $x \mapsto |x|^s$ or $x \mapsto \mathrm{sgn}(x)|x|^s$, which we parameterize by $(-1, s)$ or $(1, s)$ in $\{\pm 1\} \times \mathbb{C}$ respectively. Restricted to principal series, then, the tetrahedral symbol defines a meromorphic function on

$$\left(\{\pm 1\} \times \mathbb{C}\right)^6$$

Fixing a choice of signs, we get simply a function on $\mathbb{C}^6$. This function satisfies a *holonomic system of difference equations*, where holonomic means, roughly, that the difference equation has only a finite-dimensional space of solutions if one imposes suitable growth constraints. This follows from Proposition 6.3.1, or even more conveniently from its reformulation in (7.2.2).

Indeed, as we see from (7.2.2), what we are doing is pushing forward Lebesgue measure from (an open subset of) $\mathbb{R}^3$ to $(\mathbb{R}^\times)^6$, by means of the map

$$(x, y, z) \mapsto (x, 1 - x, x - y, y - z, z, 1 - z),$$

and taking a Mellin transform of the resulting measure $\mu$. Difference equations for that Mellin transform correspond to *differential* equations for the pushforward measure $\mu$; and that such differential equations exist in plenty follow from the fact that holonomic D-modules are stable under direct image. See [Oak18] for a very explicit discussion of such differential equations.

It would not be difficult to explicitly write down these difference equations, for example, starting with some of our hypergeometric representations. What would be particularly interesting would be to "index" the resulting system by the geometry of the spinor cone.

**9.3. The associativity kernel and analytic number theory.** The unitary integral transform indicated in Proposition 9.1.1 has played a significant role in number theory; we sketch this in a typical example. We will, for this subsection, freely assume familiarity with the language of automorphic forms.

Let us fix an anisotropic quadratic form $Q$ over a totally real[13] *global* field $F$; let $SO_3$ denote the associated orthogonal group over $F$. For each place $v$ of $F$, we let $F_v$ be the corresponding local field. Let $S$ be a finite set of places.

Now let $\pi_{12}, \pi_{23}, \pi_{34}, \pi_{41}$ be *automorphic* representations of $SO_3$, all of which are unramified outside the set $S$. For each $v \in S$, let $f_v$ be a continuous function of compact support on the tempered dual of $SO_3(F_v)$. Then we have the "associativity" formula (the terminology used in [MV10, § 4.5.2]):

$$\sum_{\pi_{24}} \frac{\mathcal{L}_2 \mathcal{L}_4}{L^{(S)}(1, \pi_{24}, \mathrm{ad})} \prod_{v \in S} f_v(\pi_{24,v}) = \sum_{\pi_{13}} \frac{\mathcal{L}_3 \mathcal{L}_1}{L^{(S)}(1, \pi_{13}, \mathrm{ad})} \prod_{v \in S} \check{f}_v(\pi_{13,v}), \tag{9.3.1}$$

[13] This enables us to avoid some irrelevant analytical issues because the Ramanujan conjecture is known.

where $f_v \leftrightarrow \check{f}_v$ is the transform of Proposition 9.1.1; the sum is taken over all automorphic representations $\pi_{\mathbf{24}}$ or $\pi_{\mathbf{13}}$, with local constituents $\pi_{\mathbf{24},v}$ and $\pi_{\mathbf{13},v}$ that are representations of $SO_3(F_v)$, and

$$\mathcal{L}_i = \sqrt{L^{(S)}\Big(\frac{1}{2}, \pi_{ij} \boxtimes \pi_{ik} \boxtimes \pi_{il}\Big)},$$

where $\{i, j, k, l\} = \mathbf{V}$, and $L^{(S)}$ means that we take L-factors only outside S. The meaning of (9.3.1) is that there exists a choice of signs for the various square roots such that it is valid. Understanding the choice of sign, from the point of view of L-functions, is a very interesting and rather unexplored question.

We sketch a proof of (9.3.1) here, at least for a class of f. Take factorizable $\psi_i \in \pi_i$ and expand

$$\int \psi_{\mathbf{1}}\psi_{\mathbf{2}}\psi_{\mathbf{3}}\psi_{\mathbf{4}} = \sum_{\pi_{\mathbf{24}}} \sum_{\psi \in \pi_{\mathbf{24}}} \int \psi_{\mathbf{1}}\psi_{\mathbf{2}}\psi \times \int \psi\psi_{\mathbf{3}}\psi_{\mathbf{4}}, \tag{9.3.2}$$

where we sum first over automorphic representations $\pi_{\mathbf{24}}$ and then over orthogonal bases for $\pi_{\mathbf{24}}$; the integrals are over the adelic quotient associated to $SO_3$. Using the Ichino–Ikeda formula (see [II10] we rewrite this as

$$c \cdot \mathcal{L}_2\mathcal{L}_4 \cdot \prod_{v \in S} \underbrace{E_{\mathbf{24}}^{(v)}(\psi_{\mathbf{1},v} \otimes \psi_{\mathbf{2},v} \otimes \psi_{\mathbf{3},v} \otimes \psi_{\mathbf{4},v})}_{f_v(\pi_{\mathbf{24},v})}$$

where c is a constant depending on measure normalization, and the other notation is as in (9.1.2), with a superscript $(v)$ to remind that we are working with $SO_3(F_v)$; note that $E_{\mathbf{24}}^{(v)}$ depends on $\pi_{\mathbf{24},v}$ and so indeed defines a function $f_v$ on the tempered dual of $SO_3(F_v)$. Now, compare this expansion with a similar expansion of (9.3.2) according to the $\{\mathbf{14}, \mathbf{23}\}$ grouping; one gets a similar structure, now involving a factor $\check{f}_v(\pi_{\mathbf{13},v})$ given by $E_{\mathbf{13}}^{(v)}(\psi_{\mathbf{1},v} \otimes \psi_{\mathbf{2},v} \otimes \psi_{\mathbf{3},v} \otimes \psi_{\mathbf{4},v})$. We conclude by Proposition 9.1.2.

**9.4. Tetrahedral and 6j symbols for $SU_2$.** There is a significant body of prior work related to this paper concerning the question of defining 6j symbols for $SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$. To our knowledge, all this work is based on definitions parallel to that of § 9.1 rather than § 3.4.3, that is to say, realizing 6j symbols as "associativity kernels," rather than in a fashion that manifestly has the symmetry of a tetrahedron. We will briefly review some of this work.

Let us first note that, although the map $SL_2 \to PGL_2$ is almost an isomorphism, the representation theory has significant differences, because of the failure of multiplicity-one: given irreducible representations $V_1$ and $V_2$ of $SL_2$ over a local field, the irreducible representations appearing in the continuous part of $V_1 \otimes V_2$ may have multiplicity 2. However, there are a number of cases related to $SL_2$ where the multiplicity one holds, in full or in part:

- The case of $SL_2(\mathbb{C})$. Here we have multiplicity-one in general ([Naĭ59]). The 6j symbols were first defined by Ismagilov [Ism06, Ism07], who gives explicit hypergeometric formulas (as a sum of the products of two ${}_4F_3$ hypergeometric series) for representations that descend to $PGL_2(\mathbb{C})$. Mellin–Barnes integrals for bona fide $SL_2(\mathbb{C})$-representations were obtained and further studied by Derkachov and Spiridonov [DS19].[14] Finally, relations to elliptic hypergeometric functions were studied in [DSS22].

- The case of $SL_2(\mathbb{R})$. Here one does not have multiplicity one in general, but it remains valid in various situations where some of the representations are discrete series. Such are the situations studied by Groenevelt [Gro03, Gro06], who expressed the relevant 6j symbols in terms of Wilson functions, which can be written as the value at 1 of certain ${}_7F_6$-function, or an equivalent form of a sum of ${}_4F_3$-functions. Note that the latter form may be compared with our results in § 7.4, and one may ask whether a similar consolidation into ${}_7F_6$-functions is available.

For some further discussion of how to address such examples within our framework, see § 9.5.3.

Genuine multiplicity two arises when tensoring two principal series of $SL_2(\mathbb{R})$. Such cases have been studied by Derkachov and Ivanov [DI23]; the definition implicitly depends on the choice of bases for various multiplicity spaces (cf. [Iva18]). It would be interesting to analyze this from the point of view of relative Langlands duality, which indeed suggests in many instances the existence of preferred bases for multiplicity spaces.

**9.5. The local Langlands correspondence.** The local Langlands correspondence permits a generalization of our prior notions of $\gamma$, $L$, and $\epsilon$-factors, which seem well adapted to the general study of the tetrahedral symbol.

Let $F$ be a local field. Attached to $F$ is a certain modification of the Galois group, the *Weil–Deligne* group $W_F$ of $F$ ([Del73, § 8.3.6]). All that is important for us now is that there is a canonical isomorphism

$$W_F^{ab} \simeq F^\times$$

and so characters of $F^\times$ can also be considered as characters of $W_F$; we will do this without comment. Now, given a representation

$$\rho\colon W_F \longrightarrow GL_n(\mathbb{C})$$

we can define $\gamma$-factor $\gamma(s,\rho)$, the $L$-function $L(s,\rho)$, and the $\epsilon$-factor $\epsilon(s,\rho)$, each meromorphic function of a complex variable $s$. In the case when $\rho$ is one-dimensional, and therefore a quasi-character of $W_F^{ab} \simeq F^\times$, they coincide with the prior definitions from § 2.2. They also satisfy a relation analogous to (2.2.3), replacing now $\chi^{-1}$ by the contragredient. In the general case, $L(s,\rho)$ always has the form $\prod_{i=1}^m L(s,\chi_i)$ for various characters $\chi_i$, and $\epsilon(s,\rho)$ always has the form $a \cdot b^s$, but $m$ can be smaller than $n$, and the values of $a, b$ are in general difficult to determine.

The significance of the representation theory of $W_F$ comes from the *local Langlands correspondence*; it asserts that there is a map $\pi \mapsto \rho_\pi$ from irreducible representations of $\mathcal{R}$,

[14] We expect that it would be straightforward to check that their formula agrees with the ones in § 7, but we have not done so.

to representations

$$\rho\colon W_F \longrightarrow SL_2(\mathbb{C}), \tag{9.5.1}$$

where $SL_2(\mathbb{C})$ is the *Langlands dual group* to $\mathcal{R}$. The representation $\rho_\pi$ is also called the *Langlands parameter* of $\pi$. For example, if $\pi$ is the principal series attached to a character $\chi$ of $F^\times$, the corresponding representation is given by $\rho = \chi \oplus \chi^{-1}$ where we identify $\chi$ with a character of $W_F$ through $W_F^{ab} \simeq F^\times$. The map that associates to $\pi$ its Langlands parameter is injective.

The tetrahedral datum $\Pi$ of § 3.3 gives rise to a parameter $\rho_\Pi\colon W_F \to SL_2^{\mathbf{E}}$, and this can be composed with the map $\tau\colon SL_2^{\mathbf{E}} \to \mathrm{Spin}_{12}$ in (4.2.1) to obtain a map

$$\rho_\Pi^{\mathrm{Spin}}\colon W_F \longrightarrow \mathrm{Spin}_{12}.$$

One aspiration that underlies much of this paper is to

*Express the theory of the tetrahedral symbol $\{\Pi\}$ entirely in terms of $\rho_\Pi^{\mathrm{Spin}}$.*

Our main theorem Theorem 5.2.1 has accomplished this in the unramified case. We will give now some further examples in this direction.

**9.5.1.** *Components on which $\{\Pi\}$ vanishes identically.* The inclusion of the center $\{\pm 1\} \hookrightarrow SL_2$ induces

$$Z \coloneqq \{\pm 1\}^{\mathbf{E}} \longrightarrow \mathrm{Spin}_{12},$$

whose image commutes with $\rho_\Pi^{\mathrm{Spin}}$. By a general construction of Gross and Prasad, explained in [GP92, § 10], we obtain from this a character

$$\psi\colon Z \to \mathbb{C}^\times,$$

namely, we associate to each $z \in Z$ the $\epsilon$-factor $\epsilon(\frac{1}{2}, S^{z=-1})$ for the action of $W_F$, acting via $\rho_\Pi^{\mathrm{Spin}}$, on the $(-1)$-eigenspace of $z \in Z$ acting on the half-spin representation.

**Lemma 9.5.2.** *If a nontrivial functional $\Lambda^H$ as in § 3.4.3 exists, $\psi$ is trivial. Conversely, if $\psi$ is trivial for a given $\rho : W_F \to SL_2^{\mathbf{E}}$, then there exists $\Pi_G$ with Langlands parameter $\rho$ (possibly after replacing $\mathcal{R}$ by the isometry group of a different quadratic form) for which $\Lambda^H$ is nontrivial.*

*Proof.* To check $\psi$ is trivial, it is enough to check that its value on the $ij$ copy of $-1$ is trivial. The $(-1)$-eigenspace for this element is given, as a representation of $SL_2^{\mathbf{E}}$, by

$$\left(\mathbb{C}^2_{ij} \otimes \mathbb{C}^2_{ik} \otimes \mathbb{C}^2_{il}\right) \oplus \left(\mathbb{C}^2_{ji} \otimes \mathbb{C}^2_{jl} \otimes \mathbb{C}^2_{jk}\right),$$

where $\{i, j, k, l\} = \mathbf{V}$. Therefore, the condition is that

$$\epsilon\Big(\frac{1}{2}, \rho_{ij} \otimes \rho_{ik} \otimes \rho_{il}\Big) = \epsilon\Big(\frac{1}{2}, \rho_{ji} \otimes \rho_{jl} \otimes \rho_{jk}\Big)$$

and since this holds for each $i, j$ we find that the value of $\epsilon(\frac{1}{2}, \rho_{ij} \otimes \rho_{ik} \otimes \rho_{il})$ is independent of $i \in \mathbf{V}$. We now apply the beautiful result of Prasad [Pra90] characterizing trilinear invariant functionals. ∎

**9.5.3.** *Generalization to* $\mathrm{SL}_2$ *from the point of view of Langlands parameters.* Let us sketch an approach, within our framework, of how to extend the tetrahedral symbol to the $\mathrm{SL}_2$ case, and how it fits with the duality formalism. The key role is played by the algebraic group

$$G^* = \mathrm{SL}_2^{\mathbf{E}}/Z',$$

where the subgroup $Z' \subset Z = (\pm 1)^{\mathbf{E}}$ of order 8 within the center of $\mathrm{SL}_2^{\mathbf{E}}$ consists of those elements whose product around each face is trivial. Equivalently, $Z'$ is *generated* by elements which are nontrivial around a given vertex

The dual group $\widetilde{G^*}$ has a remarkably similar description, but now one quotients $\mathrm{SL}_2(\mathbb{C})^{\mathbf{E}}$ now by those elements whose product around each vertex is trivial. The morphism $\tau$ does not descend to $\widetilde{G^*}$, *but* the action of $\mathrm{SL}_2(\mathbb{C})^{\mathbf{E}}$ on the half-spin $S$ does. What this means is that, given a parameter

$$W_F \to \widetilde{G^*} \tag{9.5.2}$$

one still define a representation of $W_F$ on $S$, even though one cannot define in general a $\mathrm{Spin}_{12}$-valued representation of it. A parameter (9.5.2) gives in particular an L-packet of representations of $\mathrm{SL}_2(F)$ for each edge, *such that the product of central characters around each vertex is trivial*; that is precisely the situation in which the classical 6j symbol for $\mathrm{SU}_2$ is meaningful.

It seems likely that most of the results of this paper would carry over to this situation, i.e., attach a tetrahedral symbol $\{\Pi\}$ to a datum as in (9.5.2). The embedding $H \to G$ that played a crucial role earlier is to be replaced by $H \to G^* \times G^*/Z^*$ where $Z^*$ is the antidagonal copy of the center. We have not verified the details but expect that multiplicity one holds in this context, permitting us to carry over our definitions verbatim; and it is likely that the same theorems also hold with cosmetic modifications.

**9.5.4.** *Completing Regge's original symmetry to a* $W(D_6)$*-symmetry.* The prior discussion has an interesting manifestation related to the Regge symmetries as originally envisioned by Regge. Restrict now to the case when $F = \mathbb{R}$, and let us consider the classical 6j symbol attached to the matrix $J$ of non-negative integers given by

$$J = \begin{bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix} \longmapsto \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \in \mathbb{C},$$

i.e., in our language, the tetrahedral symbol attached to the $2j+1$-dimensional representations of $\mathrm{SO}_3$. The condition for this to be defined is, with our convention, that the triangle inequalities associated to all four vertices are satisfied, e.g. $j_1, j_2, j_3$ are the lengths of a Euclidean triangle, and so forth.

The Weil group of the real numbers is an extension of $\mathbb{C}^*$ by an element $c$ (for "conjugation") satisfying $c^2 = -1$, whose action on $\mathbb{C}^*$ is given by conjugation, i.e. $czc^{-1} = \bar{z}$. The representation $\rho_j$, as in (9.5.1), associated to $V_{2j+1}$, is given by

$$\rho_j : z = re^{i\theta} \mapsto \begin{bmatrix} e^{i(2j+1)\theta} & 0 \\ 0 & e^{-i(2j+1)\theta} \end{bmatrix}, \quad c \mapsto c_{0,j} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{9.5.3}$$

Note that $\rho_j \simeq \rho_{-j-1}$.

Let us consider J as defining a class in $\tilde{\mathsf{D}}_6$, by sending each edge with positive orientation to the corresponding value of $2j+1$ (and the opposite orientation to $-2j-1$); using the isogeny $\tilde{\mathsf{D}}_6 \to \mathsf{D}_6$ we shall think of this as a cocharacter $\mathbf{J}$ for $\mathrm{Spin}_{12}$. The representation $\rho_{\mathrm{Spin}}$ is then given by

$$re^{i\theta} \mapsto \mathbf{J}(e^{i\theta}), \quad c \mapsto \dot{w}_0, \tag{9.5.4}$$

where $\dot{w}_0$ is a representative (induced by (9.5.3)) for the longest element of $W(\mathsf{D}_6)$, which negates all variables. Let us observe that any two such representatives are in fact conjugate under the maximal torus, and so the conjugacy class of (9.5.4) is independent of this choice.

Although we are no longer in the principal series case, it remains of interest to examine what symmetries of the tetrahedral symbol might be induced by $W(\mathsf{D}_6)$. If we replace $\mathbf{J}$ in (9.5.4) by $w(\mathbf{J})$ for $w \in W(\mathsf{D}_6)$, the resulting homomorphism remains conjugate to (9.5.4). The various $w(\mathbf{J})$ correspond to various collections

$$J' := \begin{bmatrix} j_1' & j_2' & j_3' \\ j_4' & j_5' & j_6' \end{bmatrix}.$$

In general, the $J'$ are only half-integral. However, they satisfy a parity constraint: the sums of $J'$s along triples of edges emanating from a single vertex are integers. In fact, this parity condition is closed under $W(\mathsf{D}_6)$-symmetry; in other words, we may relax the assumption on J to that J has half-integral entries satisfying the same parity conditions.

One checks by direct computation that, for generic J, there are 15 possibilities for $J'$ modulo $\mathfrak{I} \rtimes \mathfrak{T}$; here 15 arises from the index of $\mathfrak{I} \rtimes \mathfrak{T}$ inside $W(\mathsf{D}_6)$ (cf. Lemma 4.3.1). The various $J'$ do not correspond to homomorphisms $W_F \to \mathrm{SL}_2^{\mathsf{E}}$, but the parity properties noted above imply that they *do* correspond to homomorphisms $W_F \to \widetilde{\mathsf{G}^*}$, with notation as in § 9.5.3, and thus define L-packets of discrete series representations of $G^*(\mathbb{R})$. The tetrahedral symbol can then be defined according to the discussion of § 9.5.3.

In order to define it in all cases, however, one needs to switch between $G^*(\mathbb{R})$ and its compact form to cover all cases. Indeed, the reasoning of Lemma 9.5.2 implies the following all-or-none property for $J'$ as above: either the triangle inequalities are satisfied for all the vertices, or for none of them. In the six cases (represented by the subgroup $\mathfrak{S}_3$ in Lemma 4.3.2) where all triangle inequalities are satisfied, we interpret entries of $J'$ as indexing representations of $\mathrm{SU}_2$, and use the classical 6j symbol; in the remaining nine where no triangle inequality is satisfied, we interpret them as indexing holomorphic-antiholomorphic pairs of discrete series for $\mathrm{SL}_2(\mathbb{R})$. With this setup it becomes natural to ask:

**Question 9.5.5.** Are all of these extended classical 6j symbols for the 15 $J'$s — six attached to $\mathrm{SU}_2$ and nine attached to $\mathrm{SL}_2(\mathbb{R})$ — actually equal, up to sign and a normalizing $\gamma$-factor?

If true, this means in effect that the original Regge symmetries can indeed be "completed" to a $W(\mathsf{D}_6)$ symmetry, even though the Regge group is very much smaller. Quite possibly this question is accessible through some of our hypergeometric formulae, or through the prior work of Groenevelt [Gro03,Gro06], but we have not examined it.

**9.5.6.** *Formulas outside the principal series case.* Proposition 7.3.1 can be generalized beyond the principal series case using the Langlands formalism. Let us suppose that $\pi_{14}$ and $\pi_{23}$ are principal series, but make no supposition about the remaining $\pi$s. In the statement of Proposition 7.3.1 we can then simply replace the roles of $\rho_{ij}$ by the Langlands parameter of the representation $\pi_{ij}$. Then we anticipate that formula (7.3.1) remains valid; we have an argument for this, but have not verified the details, constants and so forth, and will return to it elsewhere.

**9.6. Functoriality and** $\mathrm{Spin}_{12}$**.** Langlands duality provides a "lifting"

$$\text{irreducible representations of } \mathcal{R}^6 \longrightarrow \text{L-packets of representation of } \mathrm{PSO}_{12}(F) \tag{9.6.1}$$

associated with the morphism $\mathrm{SL}_2^6 \to \mathrm{Spin}_{12}$ of dual groups. Our discussion of duality suggests that *the tetrahedral symbol factors through the lifting* (9.6.1).

There is a natural for the function on the right which pulls back to the tetrahedral symbol, as we now sketch. Namely, we can use the same general setup as § 8: we consider two $\mathrm{PSO}_{12}$-spaces $X, Y$ and an averaging intertwiner from $X$ to $Y$, and compute the scalar by which (9.6.2) below fails to commute:

$$\begin{tikzcd}
& C^\infty(X) \arrow[dd, dashed, "\mathrm{Av}"] \\
\Pi \arrow[ur, "\Lambda^X"] \arrow[dr, "\Lambda^Y"'] & \\
& C^\infty(Y)
\end{tikzcd} \tag{9.6.2}$$

where $\Lambda^X, \Lambda^Y$ are "normalized" embeddings of the $\mathrm{PSO}_{12}$-representation $\Pi$ into $X$ and $Y$. Our proposal is, for well-chosen $X$ and $Y$, the resulting function on (certain) $\mathrm{PSO}_{12}$-representations agrees with the tetrahedral symbol after pullback via (9.6.1).

Relative Langlands duality suggests natural choices for $X$ and $Y$: For $X$ we take the Langlands dual to the 32-dimensional (hyperspherical!) half-spin representation $S$ of $\mathrm{Spin}_{12}$; it is a generalized Bessel model for $\mathrm{PSO}_{12}$. For $Y$ we take the Whittaker model for $\mathrm{PSO}_{12}$, so its Langlands dual is a single point. Finally, to define the intertwiner Av, we average over over the smallest $G$-orbit $Z$ in $X \times Y$ that supports an invariant distribution.

In the language of § 8.2, $Z$ defines a Lagrangian

$$L \coloneqq \text{conormal of } Z \subset M \times N \coloneqq T^*X \times T^*Y,$$

whose dual $\check{L} \subset \check{M} \times \check{N} = S$ we expect to be precisely the cone of pure spinors. There is no other reasonable candidate for $\check{L}$: the preimage of $0$ under the moment map for the half-spin representation is already an irreducible Lagrangian.

**9.7. Geometric representation theory.** We will now indicate a geometrization of Theorem 5.2.1. We work now over the field $F = \overline{\mathbb{F}_p}((t))$, and write $\mathcal{O} = \overline{\mathbb{F}_p}[[t]]$. We use the same notations $G, D, H, X, Y, Z, M = T^*X, N = T^*Y, \check{M}, \check{N}$ as from § 8; we have aLagrangian $\check{L} \subset \check{M} \times \check{N}$ "induced from" the cone of pure spinors $P \subset \check{N}$.

Now, the geometric conjecture of relative Langlands duality asserts that there are equivalences of categories:

$$\text{constructible sheaves on } X_F/G_{\mathcal{O}} \sim (\text{coherent sheaves on } \check{M}/\check{G})^{\text{shear}},$$
$$\text{constructible sheaves on } Y_F/G_{\mathcal{O}} \sim (\text{coherent sheaves on } \check{N}/\check{G})^{\text{shear}}.$$

(We omit technical details about the exact categories, which can be found in [BZSV24]; also, the superscript "shear" means that the category is to be regraded as in *loc. cit.*; the details are not presently important.) Moreover, the equivalence above carries the "basic sheaf", the constant sheaf on $X_{\mathcal{O}}$ or $Y_{\mathcal{O}}$, to the structure sheaf on the coherent side, and is compatible, in a natural way, with the Satake equivalence.

The diagram $X \leftarrow Z \rightarrow Y$ gives rise, by pullback followed by pushforward, to a functor $I_{\text{aut}}$ (for "automorphic intertwiner") from constructible sheaves on $Y_F/G_{\mathcal{O}}$ to ind-constructible sheaves on $X_F/G_{\mathcal{O}}$. (Note that to actually define this requires examination of technical details that we have not carried out; the spaces $X_F, Y_F, Z_F$ are not pleasant.) A natural conjecture is that *this functor is equivalent, with respect to the equivalences above, to the push-pull* $I_{\text{spec}}$ *along the diagram*

$$\check{M}/\check{G} \longleftarrow \check{L}/\check{G} \longrightarrow \check{N}/\check{G}. \tag{9.7.1}$$

As we sketch in § 15.2, our Theorem 5.2.1, in the case of nonarchimedean $F$ of finite residue characteristic, would result from the statement by taking Frobenius trace.

**Part 2. Proofs**

## 10. Convergence and the analytic continuation

We prove all results related to convergence and analytic continuation in this section.

**10.1. Proof of Proposition 3.5.3.** Let $X$ be a smooth algebraic variety over the local field $F$ and $f_s$ a family of smooth $\mathbb{C}$-valued functions on $X(F)$ depending analytically on the complex parameter $s \in \mathbb{C}^n$, or, more generally, a parameter $s$ belonging to a complex analytic manifold.

**Definition 10.1.1.** We say $f_s$ is "a controlled family of functions on $X(F)$ depending analytically on $s$, " or for short *controlled*, if there is a smooth compactification $X \to \bar{X}$, with normal crossing boundary, with the following property: for each point $x \in \bar{X}(F)$ and $N \geqslant 0$, there exists an analytic neighbourhood $U_x \subset \bar{X}(F)$, local equations $z_1, \dots, z_r$ for the various divisorial components of the boundary passing through $x$, and an asymptotic expansion of the following form:

$$f_s = \sum_i g_{i,s} + O(|z_1 z_2 \cdots z_r|^N). \tag{10.1.1}$$

where each $g_{i,s}$ has the form

$$h_i \times |z_1|^{a_1(s)} \cdots |z_n|^{a_n(s)}$$

with $h_{i,s}$ a smooth function on $U_x$ varying meromorphically[15] in $s$, and the $a_i(s)$ analytic in $s$.

**Lemma 10.1.2.** *If $Y$ is a closed subvariety of $X$ and $f_s$ is controlled on $X$ then the restriction of $f_s$ to $Y$ is also controlled.*

*Proof.* Let $Y^*$ be the closure of $Y$ within $\bar{X}$. The boundary $Z = Y^* - Y$ is a closed set, but need not be a normal crossing divisor. Then we find an embedded resolution of singularities, i.e., a map $\pi\colon \bar{Y} \to Y^*$ with the property that $\pi^{-1}(Z)$ is a union of normal crossing divisors. In that case, the pullback of any local equation $z_i$ as above must have the form $\prod_i w_i^{b_i}$ where the $w_i$ are local equations for boundary divisors on $\bar{Y}$. From this we see that $f_s$ on $Y$ satisfies the same condition. ∎

**Lemma 10.1.3.** *Suppose $f_s$ is a controlled family of functions on $X$, and let $\omega$ be a volume form on $X$. Suppose the integral*

$$I(f_s) := \int_{X(F)} f_s |\omega|$$

*converges absolutely for some $s$. Then $I(f_s)$ can be meromorphically continued to all $s$.*

*Proof.* We analyze this by local computation, following Igusa, using the asymptotic expansion (10.1.1). Using a partition of unity reduces our statement to the meromorphicity of

$$\int_{F^n} \Phi_s(z) \times |z_1|^{a_1(s)} \cdots |z_n|^{a_n(s)}$$

where $\Phi_s(z)$ is a smooth compactly supported function of $z_1, \ldots, z_n \in F^n$ that varies holomorphically in $s$, and, by the assumed absolute convergence, there exists some value $s_0$ of $s$ for which the real part of $a_i(s_0)$ is larger than $-1$.

If $F$ is nonarchimedean, we reduce to the case when $\Phi_s$ is the characteristic function of a product of intervals $|z_i - z_{i_0}| \leqslant C$, and we leave the verification in that case to the reader.

In the archimedean case, we repeatedly integrate by parts to replace an integral by one in which all the $a_i(s)$ have real part larger than zero, and the integral becomes absolutely convergent. For example, in the case $F = \mathbb{R}$ and $n = 1$ the relevant identity is

$$\int_{\mathbb{R}} \frac{d^{2N}\phi}{dx^{2N}} |x|^{s+2N} = (s+1)\ldots(s+2N) \int \phi(x)|x|^s$$

This shows that the integral becomes holomorphic upon multiplication by a polynomial of the form $\prod_{i=1}^{n} (a_i(s)+1)(a_i(s)+2)\ldots(a_i(s)+2N)$. Our assumption on the $s_0$ shows that this polynomial does not vanish identically. ∎

[15]Note that by a "smooth function varying holomorphically," what we mean is that it is a holomorphic function of $s$ with values in the space of smooth functions with its natural topology; in the real or complex case, this is the topology induced by requiring uniform convergence of all derivatives on compact sets; $h$ is a smooth function varying *meromorphically* if $h(s)P(s)$ is holomorphic for some choice of polynomial $P$.

For us, an important example arises as follows. Suppose, to simplify the dicussion, that $F$ is nonarchimedean, and fix an open compact subgroup of $\mathcal{R}$. We need some language to be able to speak of "holomorphically varying families of vectors in holomorphic families of representations." Let $\mathcal{P}$ be the complex analytic space described in (3.5.1). By an abuse of notation, we will understand a point $\pi \in \mathcal{P}$ to index the corresponding representation if $\pi$ is discrete series, and the associated principal series representation if $\pi$ corresponds to a quasi-character of $F^\times$. With this understanding, for each $\pi \in \mathcal{P}$, the space of fixed vectors $\pi^K$ is finite-dimensional, and the various $\pi^K$ fit together to give a finite-dimensional holomorphic vector bundle over $\mathcal{P}$, the "bundle of $K$-invariants." The holomorphic structure is described as follows: positive dimensional components of $\mathcal{P}$ correspond to families of principal series representations, and the condition for a section of the vector bundle to be holomorphic is that all its point evaluations at points of $F^2 - \{0\}$ be holomorphic. Similarly, the family of *Jacquet modules* $\bar{\pi}$, i.e., the largest $U$-invariant *quotient* of the various $\pi$s, again fit together to a holomorphic vector bundle over $\mathcal{P}$, where the holomorphic structure is determined by requiring that the map $\pi^K \mapsto \bar{\pi}$ induces a holomorphic map of bundles for every $K$.

Having fixed this, we fix a self-pairing $(-,-)_\pi$ for each $\pi \in \mathcal{P}$ that varies meromorphically (i.e., it induces a meromorphic self-pairing on the vector bundle $\pi^K$ for each $K$); for example, such a pairing will be fixed in § 11. It induces, by work of Casselman, a self-pairing $(-,-)_{\bar{\pi}}$ on the Jacquet-modules $\bar{\pi}$. This "Casselman" pairing is characterized by the following property: for any $v_1, v_2 \in \pi$ we have

$$(a_x v_1, v_2)_\pi = (a_x \bar{v}_1, \bar{v}_2)_{\bar{\pi}} \tag{10.1.2}$$

for $a_x = \mathrm{diag}(x, 1)$ whenever $|x|$ is sufficiently small. In fact, for each open compact subgroup $U$ there exists $\varepsilon(U)$ so that (10.1.2) holds whenever $v_1, v_2 \in \pi^U$ and $|x| < \varepsilon(U)$. In particular, $(-,-)_{\bar{\pi}}$ also varies meromorphically. With this setup we can assert:

**Lemma 10.1.4.** *Suppose that $\pi \mapsto v_\pi$ and $\pi \mapsto w_\pi$ are holomorphic sections of the bundles of $K$-invariant vectors over $\mathcal{P}$. Then the function $\mathcal{P} \times \mathrm{PGL}_2(F) \to \mathbb{C}$, associating to $\pi \in \mathcal{P}$ and $g \in \mathrm{PGL}_2(F)$ the number $(gv_\pi, w_\pi)$, is a controlled family of functions on $\mathrm{PGL}_2(F)$ parameterized by the complex manifold $\mathcal{P}$.*

*Proof.* We use the embedding $\mathrm{PGL}_2 \to \mathbb{P}(\mathrm{Mat}_2)$. A local analytic coordinate system for a point on the boundary has the form $k_1 a_x k_2$ where $k_1, k_2$ range through open neighbourhoods in $K$, $a_x = \mathrm{diag}(x, 1)$ as above, and $|x| \leqslant c$. The desired properties follow from (10.1.2) and the following fact: for the principal series induced from $\chi$, the Jacquet module $\bar{\pi}$ is two-dimensional and the eigenvalues of $a_x$ upon it are given by $\chi_+$ and $(\chi^{-1})_+$. ∎

*Proof of Proposition 3.5.3.* For each $e \in \mathbf{E}$ choose a holomorphic family of vectors $v_\pi^{(e)}$ parameterized by $\pi \in \mathcal{P}$. Tensoring together, this induces a holomorphic family of vectors in the family of representations of $\mathcal{R}^{\mathbf{E}}$ parameterized by $\mathcal{P}^{\mathbf{E}}$.

It is enough to show that both the integrals $\int_H (hv_1, v_2) dh$ (3.4.3) and the integral $\int_{H \cap D \backslash H} \Lambda^D(hv) dh$ of (3.4.4), where $v_1, v_2, v$ are chosen to be holomorphic families of the type just described, vary meromorphically over $\mathcal{P}^{\mathbf{E}}$.

The first integral amounts to taking a family of matrix coefficients on $\mathrm{PGL}_2(F)^{\mathbf{O}}$ parameterized by $\mathcal{P}^{\mathbf{E}}$, pulling it back to $H$, and integrating. The second amounts to taking a family of matrix coefficients on $\mathrm{PGL}_2(F)^{\mathbf{O}}$, again parameterized by $\mathcal{P}^{\mathbf{E}}$, pulling it back to $(D \cap H)\backslash H \simeq (\mathrm{PGL}_2(F))^3$, and integrating. In both cases, family of matrix coefficients is controlled after pull-back by applying Lemma 10.1.4 and the desired result thereby follows from Lemma 10.1.3. ∎

**10.2. Proof of the absolute convergence in Proposition 6.3.1.** We start from the right-hand side of (6.3.6). For absolute convergence, since all $\chi_{ij}$ are unitary, it suffices to assume that they are trivial. Thus it reduces to show that $\mathcal{M}$ has finite volume with respect to the volume form

$$|\Omega|^{\frac{1}{2}} = \left| \frac{\bigwedge_e dx_e}{\sqrt{\prod_{e \neq e'} x_e \wedge x_{e'}}} \right|,$$

in which $e, e'$ are unoriented edges that share exactly one vertex. Let us recall from (6.3.4) that the statement to be proven can be put in a concrete form, namely,

$$\int_{F^3} \frac{1}{|xyz(x-y)(y-z)(1-z)(1-x)|^{\frac{1}{2}}} < \infty.$$

*First argument via high school calculus.* We only sketch this. Suppose that $F = \mathbb{R}$ for ease of visualization. We need to choose coordinates around each potential singularity of the integral and check local convergence. As an illustration, let us check conergence in $x^2+y^2+z^2 \leqslant 1$. We work in spherical coordinates. We can neglect the terms $(1-x)(1-z)$. Discarding these, the function is homogeneous of homogeneity degree $-5/2$; and since $\int_0^1 r^{-5/2} r^2 dr < \infty$ it suffices then to verify that $|xyz(x-y)(y-z)|^{-\frac{1}{2}}$ is integrable on the unit sphere. The singularities of this function lie along five great circles, and again it is enough to check local convergence. The most problematic singularity is $x = y = 0$ where three great circles meet; but in local coordinates $(u, v)$ near that point, the function looks like $|uv(u+v)|^{\frac{1}{2}}$, which is verified to be integrable by passing to polar coordinates.

*Second argument via algebraic geometry.* Algebraic geometry offers a more systematic way to write the prior type of reasoning, at the cost of requiring rather more background. Using the classical theory of moduli stack of marked curves (see [Knu83]), we can compactify $\mathcal{M}^\circ$ to a projective *variety* $\mathcal{M}$ (because we are considering genus 0 curves) whose boundary $\mathcal{M} - \mathcal{M}^\circ$ is a union of smooth divisors with normal crossings. When this is done, we have

*Claim.* $\Omega$ has at most simple poles along each boundary divisor.

The absolute convergence then follows from the simple fact that the function $|\epsilon|^{-\frac{1}{2}}$ is locally integrable near $\epsilon = 0$.

The irreducible components of the divisor $\mathcal{M} - \mathcal{M}^\circ$ can be put in three types:

(1) 2 points (i.e., $x_e$s) collide;
(2) 3 points collide in a generic fashion;
(3) 4 points collide in a generic fashion.

We leave the (easy) first case to the reader, and explain why $\Omega$ has at most a simple pole along all divisors of the other two types. It suffices in both cases to choose a coordinate chart $(a, b, \epsilon)$ near any generic point of the divisor, where $\epsilon = 0$ defines the divisor, and compute $\Omega$ in local coordinates.

In the three-point case, we take our coordinate chart to be given by

$$x_e = b + \epsilon, x_{e'} = b + a\epsilon, x_{e''} = b, \text{ remaining } xs = 0, 1, \infty,$$

where $a \neq 1$ and $b \notin \{0, 1\}$. In the four point-case we take our coordinate chart to be

$$x_e = 0, x_{e'} = \epsilon, x_{e''} = a\epsilon, x_{e'''} = b\epsilon, \text{ remaining } xs = 1, \infty,$$

where $a, b$ must avoid the loci $a = 0$, $a = 1$, $b = 0$, $b = 1$ and $a = b$. In the three-point (resp. four-point) cases, the expression for $\Omega$ is then

$$\frac{(\epsilon d\epsilon \wedge da \wedge db)^2}{\epsilon^k}, \text{ resp. } \frac{(\epsilon^2 d\epsilon \wedge da \wedge db)^2}{\epsilon^k},$$

where $k$ is the number of adjacencies amongst the edges $e, e', e''$. Evidently $k \leqslant 3$ in the three-point case. In the four point case we note that not all four of $e, e', e'', e'''$ can be simultaneously adjacent, and so $k \leqslant 5$. In both cases the order of pole of $\Omega$ is then at most 1 as desired.

**10.3. Proof of the absolute convergence of** (3.4.4)**.** We must verify that the integral of $\Lambda^D(h\nu)$ over $H \cap D \backslash H$ is absolutely convergent. Now $\Lambda^D(h\nu)$ is a certain product of matrix coefficients of tempered representation; by the results of Cowling, Haagerup and Howe [HCH88, Theorem 2], these are all majorized by a matrix coefficient of a suitable tempered principal series representation. Therefore it suffices to check the absolute convergence when all $\pi_{ij}$ are tempered principal series, i.e. principal series associated to unitary characters $\chi_{ij}$; but in that case the integral has been computed explicitly and proved to be absolutely convergent in Proposition 6.3.1 (actually, the proof of absolute convergence was was just given now, in § 10.2).

## 11. Computations with principal series

Our goal here is to prove the edge formula for principal series, stated in § 6.3. We already outlined the general plan of the computation at that point; the main issue is to be very careful about the normalizations of various functionals, because it is otherwise rather easy to compute the answer only up to an unspecified constant.

We follow the notation set up in § 6.3, so that we assign characters $\chi_{ij}$ of $F^\times$ to oriented edges such that $\chi_{ij}\chi_{ji} = 1$, and let $\pi_{ij}$ be the corresponding unitarily induced principal series representations. Let $B_{\mathcal{R}} \subset \mathcal{R} = \mathrm{PGL}_2$ be the standard upper-triangular Borel in $\mathcal{R}$ and $B = B_{\mathcal{R}}^{\mathsf{O}}$ the Borel in G.

**11.1. The plan of the computation.** Recall that $\{\Pi\}$ is defined using a D-functional on $\Pi$, denoted by $\Lambda^D$ and an H-invariant functional denoted by $\Lambda^H$. What we will do, here, is to compute with a different normalizations of such invariant functionals, which we call $\phi^D, \phi^H$, and then deduce what we want about $\Lambda^D, \Lambda^H$. Then we compare $\Lambda^H$ with another H-invariant functional obtained by averaging $\Lambda^D$:

$$(\Lambda^H)' \colon v \longmapsto \int_{H \cap D \backslash H} \Lambda^D(hv) dh.$$

Due to the technicality of this section, we give an outline of the argument for the reader's convenience:

- In § 11.2 we set up notations related to principal series.
- In §§ 11.3 and 11.4 we carry out computations related to normalizing bilinear and trilinear pairings on principal series.
- In § 11.5 we define $\phi^D$ (resp. $\phi^H$) and write $\Lambda^D$ (resp. $\Lambda^H$) in terms of it.
- in § 11.6 we compare the H-average of $\phi^D$ with $\phi^H$.
- In § 11.7 we conclude the proof, conditional on a computational lemma, which is proved in § 11.8.

**11.2. Setup on principal series.** We will summarize some essential properties and notation related to principal series for $\mathcal{R}$. In what follows, we abridge $\chi(\det g)$ to $\chi(g)$. We have fixed a local field F, and characters and measures are fixed as in § 2.1.

There is a very useful way to parameterize vectors by Schwartz functions, namely there is a natural projection $C_c^\infty(F^2) \to \pi_\chi$ where we force a function f on $F^2$ to have the desired degree of homogeneity along central directions, namely, we send

$$f(z) \longmapsto f_\chi(z) := \int_{\lambda \in F^\times} \chi^{-2}(\lambda)|\lambda| f(\lambda z) d^\times\lambda = \int_{\lambda \in F} f(\lambda z)\chi^{-2}(\lambda) d\lambda. \tag{11.2.1}$$

The normalized principal series $\pi_\chi$ is realized in the space of functions $\Phi \colon \mathcal{R} \to \mathbb{C}$ that transform on the left by means of the character

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \longmapsto (\chi^{-1})_+(a/c) = \chi\left(\frac{c}{a}\right)\left|\frac{a}{c}\right|^{\frac{1}{2}}.$$

This coincides with the definition given in § 3.5.1 in terms of functions on the punctured plane. Indeed, given $\Phi$ as above, pull it back to $F^2 - \{0\}$ by means of $(x, y) \mapsto g_{xy}$, where $g_{xy}$ is any matrix with bottom row $(x, y)$ and determinant equal to 1, and then extend it to $F^2$ by 0. The result f is independent of choice of $g_{xy}$ and satisfies $f(\lambda \cdot z) = \chi^2(\lambda)|\lambda|^{-1} f(z)$.

Now let us discuss how to rigidify this $\pi_\chi$, that is, how to endow it with a self-duality pairing. First, let us observe that on the space of $(-2)$-homogeneous functions on $F^2$ there is an action of $\mathcal{R}$ whose pullback to $GL_2$ is $g \colon f(z) \mapsto f(zg)|\det g|$, and an invariant functional given by integrating f on $\mathbb{P}^1_F$ (see § 2.4). Composing this with the product of two functions gives an invariant pairing $\pi_\chi \otimes \pi_{\chi^{-1}} \to \mathbb{C}$, which we denote simply by $(f, g) \mapsto \int_{\mathbb{P}^1} fg$.

For $z_1, z_2 \in F^2$, we denote by $z_1 \wedge z_2 \in F$ the determinant of the $2 \times 2$ matrix whose rows are respectively $z_1$ and $z_2$. Now put

$$K(z_1, z_2) = \frac{\chi^{-2}(z_1 \wedge z_2)}{|z_1 \wedge z_2|}$$

and note that $K(z_1, z_2) = K(z_2, z_1)$. Then for $f \in \pi_\chi$ the function

$$I(f)(z) := \int_{\mathbb{P}^1} K(z, z')f(z')$$

defines an element of $\pi_{\chi^{-1}}$ and the rule $f \mapsto I(f)$ intertwines $\pi_\chi$ with $\pi_{\chi^{-1}}$. We also denote $I$ by $I_\chi$ when we need to be explicit (note that $I_\chi$ and $I_{\chi^{-1}}$ are not inverse to each other). The rule

$$\langle f_1, f_2\rangle := \int_{\mathbb{P}^1} f_1 \cdot I(f_2) = \int_{\mathbb{P}^1\times\mathbb{P}^1} K(z_1, z_2)f_1(z_1)f_2(z_2)$$

defines (at least for generic $\chi$) a rigidification of $\pi_\chi$.

**11.3. The bilinear pairing on principal series.** Continue with notation as above. We have defined a rigidification of $\pi_\chi$ and $\pi_{\chi^{-1}}$. There is, up to sign, a unique isomorphism between these that preserves rigification, that we shall call the rigidified isomorphism; it is the essential uniqueness of the rigidified isomorphisms that make rigidifications so useful!

We can then transport the rigidification, either on $\pi_\chi$ or on $\pi_{\chi^{-1}}$, by means of a rigidified isomorphism in one factor, and get (up to sign) the same pairing

$$\pi_\chi \times \pi_{\chi^{-1}} \longrightarrow \mathbb{C}, \tag{11.3.1}$$

which we will call the "normalized" pairing, and which we shall compute to be

$$(f, g) \longmapsto \sqrt{\gamma(1, \chi^2)\gamma(1, \chi^{-2})} \int_{\mathbb{P}^1} fg.$$

Denote by $\mathcal{S}(F^2)$ the Schwartz space on $F^2$, namely the space of smooth rapidly decreasing functions if $F$ is archimedean, or locally constant functions with compact support if $F$ is nonarchimedean. We normalize the Fourier transform $\mathcal{F}_2$ on $F^2$ by means of

$$\mathcal{F}_2\Phi(z_1) = \int_{z_2\in F^2} \Psi(z_1 \wedge z_2)\Phi(z_2). \tag{11.3.2}$$

Note that since we use a *skew-symmetric* pairing inside $\Psi$ instead of a symmetric one, one has $\mathcal{F}_2^2 = \mathrm{id}$ without also negating $z$ (cf. § 2.3).

**Lemma 11.3.1** (Intertwiner versus Fourier transform)**.** *The following diagram commutes:*

$$\begin{tikzcd} \mathcal{S}(F^2) \arrow[r, "\mathcal{F}_2"] \arrow[d, "f\mapsto f_\chi"'] & \mathcal{S}(F^2) \arrow[d, "f\mapsto f_{\chi^{-1}}"] \\ \pi_\chi \arrow[r, "\gamma(1,\chi^2)^{-1}I_\chi"] & \pi_{\chi^{-1}} \end{tikzcd}$$

*Moreover, $I_{\chi^{-1}}I_\chi$ is the same as multiplication by $\gamma(1, \chi^2)\gamma(1, \chi^{-2})$.*

*Proof.* We compute

$$\begin{aligned} I_\chi(f_\chi)(z) &= \int_{\lambda\in F, z'\in\mathbb{P}^1} \chi^{-2}(\lambda)f(\lambda z')\chi_+^{-2}(z \wedge z')dz'd\lambda \\ &= \int_{(x,y)\in F^2} f(x, y)\chi_+^{-2}\big(z \wedge (x, y)\big)dxdy. \end{aligned}$$

On the other hand, we also have

$$\mathcal{F}_2(f)_{\chi^{-1}}(z) = \int_{\lambda\in F, z'\in F^2} \chi^2(\lambda)\Psi(\lambda z \wedge z')f(z')dz'd\lambda = \gamma(1,\chi^2)^{-1} I_\chi(f_\chi)(z),$$

where we first carried out $\lambda$-integral to give $\gamma(1,\chi^2)^{-1}\chi_+^{-2}(z\wedge z')$. Here we used from (2.2.5) the fact that the Fourier transform of a character $\chi$, considered as a function on the additive group, is $(\chi(-1)\gamma(1,\chi)\chi_1)^{-1}$ (and that $\chi^2(-1)=1$). This proves the commutativity of the diagram.

Moreover, applying the above equation with $\chi$ replaced by $\chi^{-1}$ and $f$ by $\mathcal{F}_2(f)$ we get

$$\mathcal{F}_2^2(f)_\chi = \gamma(1,\chi^{-2})^{-1} I_{\chi^{-1}}\left(\mathcal{F}_2(f)_{\chi^{-1}}\right) = \gamma(1,\chi^{-2})^{-1}\gamma(1,\chi^2)^{-1} I_{\chi^{-1}} I_\chi(f_\chi), \tag{11.3.3}$$

from which we obtain the second claim. ∎

**Lemma 11.3.2.** *A* rigidified *isomorphism*

$$\pi_\chi \boxtimes \pi_{\chi^{-1}} \longrightarrow \pi_\chi \boxtimes \pi_\chi$$

*is given by* $\gamma(1,\chi^2)^{-\frac12}\gamma(1,\chi^{-2})^{-\frac12}(\mathrm{id}\otimes I_{\chi^{-1}})$. *Correspondingly, a normalized pairing on* $\pi_\chi \boxtimes \pi_{\chi^{-1}}$ *(as in* (11.3.1)*) is given by*

$$(f,g) \longmapsto \sqrt{\gamma(1,\chi^2)\gamma(1,\chi^{-2})} \int_{\mathbb{P}^1} fg.$$

*Proof.* Use $\nu,\nu'$ to denote for vectors in $\pi_\chi$ and $w,w'$ for vectors in $\pi_{\chi^{-1}}$. Now, integration against the kernel $K(x,y)$ above, or its analogue for $\chi^{-1}$, give respectively

$$I_\chi : \pi_\chi \longrightarrow \pi_{\chi^{-1}},$$
$$I_{\chi^{-1}} : \pi_{\chi^{-1}} \longrightarrow \pi_\chi.$$

The self-duality structures are given by $\int_{\mathbb{P}^1} \nu \cdot I_\chi(\nu')$ on $\pi_\chi$ and similarly $\int_{\mathbb{P}^1} w \cdot I_{\chi^{-1}}(w')$ on $\pi_{\chi^{-1}}$. Consider now

$$\begin{aligned} \mathrm{id}\otimes I_{\chi^{-1}} : \pi_\chi \boxtimes \pi_{\chi^{-1}} &\longrightarrow \pi_\chi \boxtimes \pi_\chi \\ \nu\otimes w &\longmapsto \nu \otimes I_{\chi^{-1}}(w). \end{aligned}$$

Transporting back the self-duality form $\int_{\mathbb{P}^1} \nu_1 \cdot I_\chi(\nu_1') \times \int_{\mathbb{P}^1} \nu_2 \cdot I_\chi(\nu_2')$ on the right hand side, we get on the left the self-duality form

$$\begin{aligned} &\int_{\mathbb{P}^1} \nu \cdot I_\chi(\nu') \times \int_{\mathbb{P}^1} I_{\chi^{-1}}(w) \cdot I_\chi I_{\chi^{-1}}(w') \\ &\qquad \overset{(11.3.3)}{=} \int_{\mathbb{P}^1} \nu \cdot I_\chi(\nu') \times \int_{\mathbb{P}^1} I_{\chi^{-1}}(w)\cdot w' \cdot \gamma(1,\chi^2)\gamma(1,\chi^{-2}). \end{aligned}$$

Since $K$ is symmetric in its two arguments, $I_{\chi^{-1}}$ is adjoint to $I_\chi$ with respect to the pairing $\int_{\mathbb{P}^1}$, and so we can rewrite the above as $\gamma(1,\chi^2)\gamma(1,\chi^{-2})$ multiplied by the standard self-duaity pairing on $\pi_\chi \boxtimes \pi_{\chi^{-1}}$. The first claim follows, and the second claim is a consequence of the first. ∎

**11.4. Normalizing the trilinear functional.** Now fix three characters $\chi_1, \chi_2, \chi_3$ with corresponding principal series representations $\pi_{\chi_i}$, and write $H'$ for the diagonal copy of $\mathcal{R}$ inside $G' = \mathcal{R}^3$. Put

$$\Pi' = \pi_{\chi_1} \boxtimes \pi_{\chi_2} \boxtimes \pi_{\chi_3}.$$

We also let $\chi_{123} = \chi_1\chi_2\chi_3$.

We rigidify each $\pi_{\chi_i}$ and so also $\Pi'$ according to the discussion above. Our interest here will be to compute the normalized (in the sense of (3.4.3)) $H'$-invariant functional on the space of $\Pi'$. The open orbit of $H'$ on $(\mathbb{P}^1)^3$ is the locus $O$ where the three points are distinct, which we denote by $[x_j, y_j]$ for $j = 1, 2, 3$. We fix a basepoint $p_0 \in O$ and a lift $\tilde{p}_0 \in (F^2)^3$; for concreteness we take $p_0 = (0, 1, \infty)$ and

$$\tilde{p}_0 = \big((0,1),(1,1),(1,0)\big) \in (F^2)^3.$$

The $H'$-invariant measure is induced by the 3-form on $(\mathbb{P}^1)^3$ whose pullback to $(F^2 - \{0\})^3$ is

$$dp = \frac{\prod_{j=1}^3 (x_j dy_j - y_j dx_j)}{(x_1y_2 - x_2y_1)(x_2y_3 - x_3y_2)(x_3y_1 - x_1y_3)}. \tag{11.4.1}$$

The choice of order in the product does not matter, because the measure induced by a differential form does not depend on sign. It is easily seen that this formula is independent of the choices of $x_j$ and $y_j$, and is invariant under $H'$. We transport this measure to $H'$ by means of $h \in H' \mapsto p_0 h$. The resulting measure $d^{\mathbb{P}}h$ is independent of the choice of $p_0$. We remind the reader that if $F$ is nonarchimedean, this measure assigns $H'(\mathcal{O}) = \mathcal{R}(\mathcal{O})$ volume $1 - q^{-2}$, hence differs from the normalized Haar measure (cf. § 3.2). Evidently, the following form on $\Pi'$ is $H'$-invariant:

$$\tau(f_1 \boxtimes f_2 \boxtimes f_3) = \int_{H'} (hf_1 \boxtimes hf_2 \boxtimes hf_3)(\tilde{p}_0) d^{\mathbb{P}}h = \int_{H'} \psi(\tilde{h}) f(\tilde{p}_0 \tilde{h}) d^{\mathbb{P}}h, \tag{11.4.2}$$

where we choose some $\tilde{h} \in \mathrm{GL}_2$ lifting $h$, and $\psi = (\chi_{123}^{-1}|\cdot|^{3/2}) \circ \det$.

**Lemma 11.4.1.** *With the choice of $\tilde{p}_0$ above, we have*

$$\tau^{\text{norm}} = \alpha^{\frac{1}{2}} \tau,$$

*where $\tau^{\text{norm}}$ is the "normalized" trilinear functional on $\Pi'$ defined via $\tau^{\text{norm}}(v_1)\tau^{\text{norm}}(v_2) = \int_{h \in H'} (hv_1, v_2) dh$, and*

$$\alpha = \nu_{\mathbb{P}}^{-1} \prod_{i=1}^{3} \gamma(1, \chi_i^2) \cdot \gamma\Big(\frac{1}{2}, \chi_{\bar{1}\bar{2}\bar{3}}\Big) \gamma\Big(\frac{1}{2}, \chi_{1\bar{2}\bar{3}}\Big) \gamma\Big(\frac{1}{2}, \chi_{\bar{1}2\bar{3}}\Big) \gamma\Big(\frac{1}{2}, \chi_{\bar{1}\bar{2}3}\Big).$$

*where we use the shorthand $\chi_{1\bar{2}\bar{3}}$ for $\chi_1\chi_2^{-1}\chi_3^{-1}$, and so on, and $\nu_{\mathbb{P}}$ is defined by* (3.2.3).

*Proof.* First, note that

$$\tau(f) = \int_{F^3} f_1(x_1,1) f_2(x_2,1) f_3(x_3,1) \frac{\chi_{1\bar{2}\bar{3}}(x_2 - x_3)\chi_{\bar{1}2\bar{3}}(x_3 - x_1)\chi_{\bar{1}\bar{2}3}(x_1 - x_2)}{|(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)|^{\frac{1}{2}}} dx_1 dx_2 dx_3. \tag{11.4.3}$$

To see this we set

$$\tilde{h} = \begin{bmatrix} x_3 b & b \\ x_1 d & d \end{bmatrix} \tag{11.4.4}$$

with $b = (x_1 - x_2)$ and $d = (x_2 - x_3)$. Note that $h$ maps $p_0$ to $(x_1, x_2, x_3)$; in fact,

$$\tilde{p}_0 \tilde{h} = ((x_2 - x_3)[x_1, 1], -(x_3 - x_1)[x_2, 1], (x_1 - x_2)[x_3, 1]) \in (\mathbb{P}^1)^3.$$

Note also that $\det \tilde{h} = (x_2 - x_3)(x_3 - x_1)(x_1 - x_2)$. Using these coordinates for $h$ in (11.4.2), the claim follows.

Now, take $f \in \Pi'$ and $g \in \tilde{\Pi}' := \pi_{\chi_1^{-1}} \boxtimes \pi_{\chi_2^{-1}} \boxtimes \pi_{\chi_3^{-1}}$. Then we have

$$\begin{aligned} \tau(f)\tau(g) &= \int_{h_1, h_2 \in H'} \psi(\tilde{h}_1)\psi'(\tilde{h}_2) f(\tilde{p}_0 \tilde{h}_1) g(\tilde{p}_0 \tilde{h}_2) d^{\mathbb{P}} h_1 d^{\mathbb{P}} h_2 \\ &= \int_{x, h_2 \in H'} (xf)(\tilde{p}_0 \tilde{h}_2) g(\tilde{p}_0 \tilde{h}_2) |\det(\tilde{h}_2)|^3 d^{\mathbb{P}} x d^{\mathbb{P}} h_2, \end{aligned} \tag{11.4.5}$$

where we substituted $x = \tilde{h}_2^{-1}\tilde{h}_1$, $\psi = (\chi_{123}^{-1}|\cdot|^{3/2}) \circ \det$ and $\psi' = (\chi_{123}|\cdot|^{3/2}) \circ \det$. The product $\phi := (xf) \cdot g$ is a function of homogeneous degree $-2$ and for such a function we have

$$\int_{H'} \phi(\tilde{p}_0 \tilde{h}) |\det(\tilde{h})|^3 d^{\mathbb{P}} h = \int_{(\mathbb{P}^1)^3} \phi. \tag{11.4.6}$$

To check (11.4.6) we express both as integrals over $F^3$. The right hand side equals $\int_{F^3} \phi(x_1, x_2, x_3) dx_1 dx_2 dx_3$ by (2.4.1). On the left hand side, we parametrize elements $\tilde{h}$ by means of $(x_1, x_2, x_3)$ as in (11.4.4). The function $p_0 h \mapsto \phi(\tilde{p}_0 \tilde{h})|\det(\tilde{h})|^3$, considered as a function on the orbit $O$, therefore assigns to $(x_1, x_2, x_3) \in F^3 \subset \mathbb{P}^1(F)^3$ the value $\Delta\phi(x_1, x_2, x_3)$, where $\Delta = |(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)|$, and therefore its integral against the measure $\Delta^{-1} dx_1 dx_2 dx_3$ (cf. (11.4.1)) coincides with the right hand side of (11.4.6). That concludes the proof of (11.4.6).

Combining (11.4.5) with (11.4.6) we find that $\tau(f)\tau(g)$ coincides with the integral of the pairing

$$\int_{x \in H'} d^{\mathbb{P}} x \int_{(\mathbb{P}^1)^3} (xf) \cdot g = \nu_{\mathbb{P}} \int_{x \in H'} dx \int_{(\mathbb{P}^1)^3} (xf) \cdot g.$$

Now, recall that the normalized pairing on $\pi_\chi$ is defined by $\langle f, g \rangle = \int_{\mathbb{P}^1} f \cdot I(g)$; we therefore obtain for $f, g \in \Pi'$ the equality

$$\tau^{\text{norm}}(f)\tau^{\text{norm}}(g) = \int_{H'} dx \int_{(\mathbb{P}^1)^3} xf \cdot I(g) = \tau(f)\tau(I(g)),$$

where $I$ now connotes the product of intertwining operators in each of the three tensor variables. It remains now to verify that

$$\tau(I(g)) = \alpha\tau(g).$$

To prove this, we may suppose that $g = \boxtimes_i \Phi_{i,\chi_i}$ for some $\Phi_i \in C_c^\infty(F^2)$ (cf. (11.2.1)), and then from (11.4.3)

$$\tau(g) = \int_{(z_1,z_2,z_3)\in(F^2)^3} \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\frac{\chi_{1\bar{2}\bar{3}}(z_2\wedge z_3)\chi_{\bar{1}2\bar{3}}(z_3\wedge z_1)\chi_{\bar{1}\bar{2}3}(z_1\wedge z_2)}{|(z_2\wedge z_3)(z_3\wedge z_1)(z_1\wedge z_2)|^{\frac{1}{2}}}.$$

Using Lemma 11.3.1, we have

$$\begin{aligned}\tau(I(g)) &= \prod_i \gamma(1,\chi_i^2) \\ &\quad\times \int_{(z_1,z_2,z_3)} \mathcal{F}_2(\Phi_1\boxtimes\Phi_2\boxtimes\Phi_3)\frac{\chi_{\bar{1}23}(z_2\wedge z_3)\chi_{1\bar{2}3}(z_3\wedge z_1)\chi_{12\bar{3}}(z_1\wedge z_2)}{|(z_2\wedge z_3)(z_3\wedge z_1)(z_1\wedge z_2)|^{\frac{1}{2}}},\end{aligned} \tag{11.4.7}$$

where we still use $\mathcal{F}_2$ to denote the Fourier transform on $(F^2)^3$ induced by $\mathcal{F}_2$ in (11.3.2). For $\Phi \in C_c^\infty((F^2)^3)$ and a tempered distribution $\chi$ on $(F^2)^3$, adjointness of the Fourier transform gives

$$\int_{(F^2)^3} \Phi\cdot\bar{\chi} = \int_{(F^2)^3} \mathcal{F}_2(\Phi)\cdot\overline{\mathcal{F}_2(\chi)},$$

where $\bar{\chi}$ means the complex conjugation.

For additive character $\Psi$, $\overline{\Psi(x)} = \Psi(x)^{-1} = \Psi(-x)$, we have then $\mathcal{F}_2(\bar{\chi})(y) = \overline{\mathcal{F}_2(\chi)}(-y)$. As a result, (11.4.7) is equal to

$$\prod_i \gamma(1,\chi_i^2)\int_{(z_1,z_2,z_3)\in(F^2)^3} \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\cdot\mathcal{F}_2(\chi)(-z_1,-z_2,-z_3),$$

where $\chi$ is the distribution given by

$$(z_1,z_2,z_3) \longmapsto \frac{\chi_{\bar{1}23}(z_2\wedge z_3)\chi_{1\bar{2}3}(z_3\wedge z_1)\chi_{12\bar{3}}(z_1\wedge z_2)}{|(z_2\wedge z_3)(z_3\wedge z_1)(z_1\wedge z_2)|^{\frac{1}{2}}}.$$

The result then follows from Lemma 11.4.2 below. ∎

**Lemma 11.4.2.** *Let $\alpha_1,\alpha_2,\alpha_3$ be characters of $F^\times$ and $\alpha_{123} = \alpha_1\alpha_2\alpha_3$. Then the distribution on $(F^2)^3$ given by $\alpha_1(z_2\wedge z_3)\alpha_2(z_3\wedge z_1)\alpha_3(z_1\wedge z_2)$ has Fourier transform*

$$\Big(\gamma(1,\alpha_{123}|\cdot|)\prod_i\gamma(1,\alpha_i)\Big)^{-1}\times\frac{\alpha_1^{-1}(z_2\wedge z_3)\alpha_2^{-1}(z_3\wedge z_1)\alpha_3^{-1}(z_1\wedge z_2)}{|(z_2\wedge z_3)(z_3\wedge z_1)(z_1\wedge z_2)|}.$$

*Proof.* We will proceed formally, leaving the routine analysis details of dealing with distributions (in contrast to functions) to the reader. The basic strategy is simply to carry out the Fourier transform first in the $z_1$ variable, then $z_2$, then $z_3$; each of these will be straightforward after a change of coordinates.

Write $z_i = (x_i, y_i)$ and $dz_i = dx_i\wedge dy_i$. Using the equalities

$$\begin{aligned}\big((z_2\wedge z_3)dz_1\big)\wedge dz_2\wedge dz_3 &= \big(d(z_2\wedge z_1)\wedge d(z_3\wedge z_1)\big)\wedge dz_2\wedge dz_3,\\ z_1 &= \frac{(z_1\wedge z_3)z_2-(z_1\wedge z_2)z_3}{z_2\wedge z_3},\end{aligned}$$

we carry out the first change of coordinates, fixing $z_2$ and $z_3$ but replacing $z_1$ by

$$z_1' = (x_1', y_1'), \quad x_1' = z_2 \wedge z_1, y_1' = z_3 \wedge z_1.$$

Perform Fourier transformation with respect to $z_1$ with $k_1$ being the dual coordinate of $z_1$, in other words, we compute the integral

$$\begin{aligned}
&\int_{F^2} \Psi(k_1 \wedge z_1)\alpha_1(z_2 \wedge z_3)\alpha_2(z_3 \wedge z_1)\alpha_3(z_1 \wedge z_2)dz_1 \\
&= \int_{F^2} \Psi\Big(\frac{k_1 \wedge z_2}{z_2 \wedge z_3}(-y_1') + \frac{k_1 \wedge z_3}{z_2 \wedge z_3}x_1'\Big)\alpha_1(z_2 \wedge z_3)\alpha_2(y_1')\alpha_3(-x_1')\frac{dz_1'}{|z_2 \wedge z_3|} \\
&= \frac{\alpha_1(z_2 \wedge z_3)}{|z_2 \wedge z_3|}\int_F \Psi\Big(\frac{k_1 \wedge z_2}{z_2 \wedge z_3}(-y_1')\Big)\alpha_2(y_1')dy_1' \int_F \Psi\Big(\frac{k_1 \wedge z_3}{z_2 \wedge z_3}x_1'\Big)\alpha_3(-x_1')dx_1' \\
&= \alpha_{23}(-1)\frac{\alpha_1(z_2 \wedge z_3)}{|z_2 \wedge z_3|}\mathcal{F}(\alpha_3)\Big(\frac{k_1 \wedge z_3}{z_2 \wedge z_3}\Big)\mathcal{F}(\alpha_2)\Big(\frac{k_1 \wedge z_2}{z_2 \wedge z_3}\Big) \\
&\overset{(2.2.5)}{=} \big(\gamma(1, \alpha_2)\gamma(1, \alpha_3)\big)^{-1}\alpha_{\bar{3};-1}(k_1 \wedge z_3)\alpha_{\bar{2};-1}(k_1 \wedge z_2)\alpha_{123;1}(z_2 \wedge z_3),
\end{aligned}$$

where we use a semicolon to separate the indexing subscripts $(1, 2, 3, 123$, etc.$)$ from the twisting subscripts $(\pm 1)$.

Similarly, we write

$$z_2 = \frac{(z_2 \wedge z_3)k_1 - (z_2 \wedge k_1)z_3}{k_1 \wedge z_3},$$

and transform with respect to $z_2$. We get

$$\begin{aligned}
&\int_{F^2} \Psi(k_2 \wedge z_2)\alpha_{\bar{3};-1}(k_1 \wedge z_3)\alpha_{\bar{2};-1}(k_1 \wedge z_2)\alpha_{123;1}(z_2 \wedge z_3)dz_2 \\
&= \frac{\alpha_{\bar{3};-1}(k_1 \wedge z_3)}{|k_1 \wedge z_3|}\mathcal{F}(\alpha_{123;1})\Big(\frac{k_2 \wedge k_1}{k_1 \wedge z_3}\Big)\mathcal{F}(\alpha_{\bar{2};-1})\Big(\frac{k_2 \wedge z_3}{k_1 \wedge z_3}\Big) \\
&= \alpha_{13}(-1)\big(\gamma(1, \alpha_{123;1})\gamma(1, \alpha_{\bar{2};-1})\big)^{-1}\alpha_{\bar{1}\bar{2}\bar{3};-2}(k_2 \wedge k_1)\alpha_2(k_2 \wedge z_3)\alpha_1(k_1 \wedge z_3).
\end{aligned}$$

Finally, we now write

$$z_3 = \frac{(z_3 \wedge k_2)k_1 - (z_3 \wedge k_1)k_2}{k_1 \wedge k_2},$$

and transform with respect to $z_3$. We get

$$\begin{aligned}
&\int_{F^2} \Psi(k_3 \wedge z_3)\alpha_{\bar{1}\bar{2}\bar{3};-2}(k_2 \wedge k_1)\alpha_2(k_2 \wedge z_3)\alpha_1(k_1 \wedge z_3) \\
&= \alpha_2(-1)\frac{\alpha_{\bar{1}\bar{2}\bar{3};-2}(k_2 \wedge k_1)}{|k_1 \wedge k_2|}\mathcal{F}(\alpha_2)\Big(\frac{k_3 \wedge k_1}{k_1 \wedge k_2}\Big)\mathcal{F}(\alpha_1)\Big(\frac{k_3 \wedge k_2}{k_1 \wedge k_2}\Big) \\
&= \alpha_{23}(-1)\big(\gamma(1, \alpha_2)\gamma(1, \alpha_1)\big)^{-1}\alpha_{\bar{2};-1}(k_3 \wedge k_1)\alpha_{\bar{1};-1}(k_3 \wedge k_2)\alpha_{\bar{3};-1}(k_1 \wedge k_2) \\
&= \alpha_{123}(-1)\big(\gamma(1, \alpha_2)\gamma(1, \alpha_1)\big)^{-1}\alpha_{\bar{1};-1}(k_2 \wedge k_3)\alpha_{\bar{2};-1}(k_3 \wedge k_1)\alpha_{\bar{3};-1}(k_1 \wedge k_2).
\end{aligned}$$

Combining three steps together, the Fourier transform on $(F^2)^3$ of the distribution $\alpha_1(z_2 \wedge z_3)\alpha_2(z_3 \wedge z_1)\alpha_3(z_1 \wedge z_2)$ is then

$$\alpha_{\bar{1};-1}(k_2 \wedge k_3)\alpha_{\bar{2};-1}(k_3 \wedge k_1)\alpha_{\bar{3};-1}(k_1 \wedge k_2)$$

times

$$\alpha_2(-1)\big(\gamma(1,\alpha_2)\gamma(1,\alpha_3)\gamma(1,\alpha_{123;1})\gamma(1,\alpha_{\bar{2};-1})\gamma(1,\alpha_1)\gamma(1,\alpha_2)\big)^{-1}.$$

Notice that $\gamma(1,\alpha_2)\gamma(1,\alpha_{\bar{2};-1}) = \alpha_2(-1)$, and so we see that the Fourier transform of $\alpha_1(z_2 \wedge z_3)\alpha_2(z_3 \wedge z_1)\alpha_3(z_1 \wedge z_2)$ is precisely

$$\Big(\gamma(1,\alpha_{123;1})\prod_i \gamma(1,\alpha_i)\Big)^{-1}\alpha_{\bar{1};-1}(k_2 \wedge k_3)\alpha_{\bar{2};-1}(k_3 \wedge k_1)\alpha_{\bar{3};-1}(k_1 \wedge k_2),$$

as desired. ∎

**11.5. Definition of $\phi^D, \Lambda^D$ and $\phi^H, \Lambda^H$.** We now follow the notation of § 6.4, and the reader may find the outline there helpful before reading this and the next subsections.

We define the naive functional $\phi^D$ on $\Pi$ so that its value on the vector $f = \otimes_{i \neq j} f_{ij}$ is given by the formula:

$$\phi^D(f) = \prod_{ij \in \mathbf{E}} \int_{\mathbb{P}^1(F)} f_{ij}(z_{ij}) f_{ji}(z_{ij}) dz_{ij},$$

where $z_{ij} = (x_{ij}, y_{ij})$ is any representative of the homogeneous coordinate $[x_{ij}, y_{ij}]$ on $\mathbb{P}^1$, and denote $dz_{ij} = x_{ij}dy_{ij} - y_{ij}dx_{ij}$ (note that this notation is different from the previous section where $dz = dx \wedge dy$). By Lemma 11.3.2 we obtain the formula for the normalized pairing $\Lambda^D$:

$$\Lambda^D = \phi^D \cdot \prod_{ij \in \mathbf{O}} \gamma(1, \chi_{ij}^2)^{\frac{1}{2}}. \tag{11.5.1}$$

Next, we construct an H-invariant functional $\phi^H$, annd relate it to the normalized functional $\Lambda^H$. The H-orbits on $B\backslash G = (\mathbb{P}^1)^{\mathbf{O}}$ can be described by looking at the triads pointing outwards (or equivalently, inwards) from a given vertex. There is a unique open dense $\mathcal{R}$-orbit on the product of three copies of $\mathbb{P}^1$ indexed by each triad; we denote this open orbit corresponding to vertex $i$ by $O_i$. The Haar measure on $\mathcal{R}$ induces an invariant measure on $O_i$, which has been described in (11.4.1).

Fix a base point $p_0 = (p_{ij})_{ij \in \mathbf{O}}$ in $\prod_i O_i \subset (\mathbb{P}^1)^{\mathbf{O}}$. and a lifting $\tilde{p}_0 = (\tilde{p}_{ij}) \in (\mathbb{A}^2)^{\mathbf{O}}$. To interface with the computation in Lemma 11.4.1, we make our choices as follows:

$$\tilde{p}_{\mathbf{12}} = \tilde{p}_{\mathbf{21}} = \tilde{p}_{\mathbf{34}} = \tilde{p}_{\mathbf{43}} = (0,1),$$
$$\tilde{p}_{\mathbf{13}} = \tilde{p}_{\mathbf{31}} = \tilde{p}_{\mathbf{24}} = \tilde{p}_{\mathbf{42}} = (1,1),$$
$$\tilde{p}_{\mathbf{14}} = \tilde{p}_{\mathbf{41}} = \tilde{p}_{\mathbf{23}} = \tilde{p}_{\mathbf{32}} = (1,0).$$

In words, nonadjacent pair of edges (i.e., edges not sharing any vertex) regardless of orientation (e.g. **12**, **21**, **34**, **43**) are all assigned the same point; these points are $0, 1, \infty$ and their "standard" lifts to $F^2$.

With this choice of $\tilde{p}_0$, we define $\phi^H$ as the product of the previously defined $\tau$-functionals from (11.4.2) on the various triads of representations; symbolically,

$$\phi^H(f) = \prod_i \int_H f_{ij}(\tilde{p}_{ij}\tilde{g}_i)\psi_i(\tilde{g}_i)d^{\mathbb{P}}g_i \tag{11.5.2}$$

where $\psi_i = (\chi_{ij}\chi_{ik}\chi_{il})^{-1}|\cdot|^{3/2}$. By Lemma 11.4.1, we have also

$$\begin{aligned}\Lambda^H &= \nu_{\mathbb{P}}^{-2}\phi^H \times \Bigg(\prod_{ij\in\mathbf{O}}\gamma(1,\chi_{ij}^2)\cdot\prod_{i\in\mathbf{V}}\gamma\Big(\frac{1}{2},\chi_{ij}^{-1}\chi_{ik}^{-1}\chi_{il}^{-1}\Big)\gamma\Big(\frac{1}{2},\chi_{ij}\chi_{ik}^{-1}\chi_{il}^{-1}\Big)\\ &\qquad\qquad \gamma\Big(\frac{1}{2},\chi_{ij}^{-1}\chi_{ik}\chi_{il}^{-1}\Big)\gamma\Big(\frac{1}{2},\chi_{ij}^{-1}\chi_{ik}^{-1}\chi_{il}\Big)\Bigg)^{\frac{1}{2}}\\ &= \nu_{\mathbb{P}}^{-2}\phi^H \times \Bigg(\prod_{ij\in\mathbf{O}}\gamma(1,\chi_{ij}^2)\cdot\gamma\Big(\frac{1}{2},S^-\Big)\Bigg)^{\frac{1}{2}}.\end{aligned} \tag{11.5.3}$$

**11.6. Averaging.** Again, the reader may refer to the outline in § 6.4. First, the $H$-average of $\phi^D$, denoted by $(\phi^H)'$, is given by

$$(\phi^H)'(f) = \int_{[g_i]\in\mathcal{R}\backslash H}\Bigg[\prod_{ij\in\mathbf{E}}\int_{\mathbb{P}^1}\underbrace{\chi_{ij-}^{-1}(\tilde{g}_i)\chi_{ij-}^{-1}(\tilde{g}_j)f_{ij}(z_{ij}\tilde{g}_i)f_{ji}(z_{ij}\tilde{g}_j)}_{P_{ij}}\,dz_{ij}\Bigg]d[g_i],$$

where we have written $\chi_{ij-}^{-1}(\tilde{g})$ as an abbreviation of $\chi_{ij-}^{-1}(\det\tilde{g})$; the function $P_{ij}$ is $(-2)$-homogeneous in coordinate $z_{ij}$. In other words $(\phi^H)'$ is the integral of the 15-form

$$\omega_1 = \frac{\prod_{ij\in\mathbf{E}}P_{ij}\cdot dz_{ij}\prod_{i\in\mathbf{V}}dg_i}{dg}, \tag{11.6.1}$$

over the space

$$\mathcal{R}\backslash\big((\mathbb{P}^1)^{\mathbf{E}}\times H\big), \tag{11.6.2}$$

where $\mathcal{R}$ acts on each $\mathbb{P}^1$-factor by $h\colon [x,y]\mapsto[x,y]h^{-1}$, and where $dg$ corresponds to the Haar measure on the diagonal $\mathcal{R}$.

On the other hand the integral (11.5.2) $\phi^H$ can be rewritten as an integral of a 12-form

$$\omega_2 = \prod_{ij\in\mathbf{O}}Q_{ij}\cdot\prod_{i\in\mathbf{V}}d^{\mathbb{P}}g_i = \nu_{\mathbb{P}}^4\prod_{ij\in\mathbf{O}}Q_{ij}\cdot\prod_{i\in\mathbf{V}}dg_i \tag{11.6.3}$$

over $H$, where $Q_{ij} = f_{ij}(\tilde{p}_{ij}\tilde{g}_i)\chi_{ij-}^{-1}(\tilde{g}_i)$ and $\nu_{\mathbb{P}}$ is as in (3.2.3).

To relate these, let us first relate more carefully the spaces over which we are integrating. There is a natural map

$$\begin{aligned}A\colon (\mathbb{P}^1)^{\mathbf{E}}\times H &\longrightarrow (\mathbb{P}^1)^{\mathbf{O}},\\ (z_{ij}=z_{ji},g_i) &\longmapsto (z_{ij}g_i).\end{aligned}$$

It is $\mathcal{R}$-invariant where $\mathcal{R}$ is acting on $(\mathbb{P}^1)^{\mathbf{E}} \times H$ as in (11.6.2). We examine its restriction over the open subset $O = \prod_i O_i$, and descend it to the quotient of the domain by $\mathcal{R}$:

$$\bar{A}\colon \mathcal{R}\backslash A^{-1}(O) \longrightarrow O \simeq H,$$

where the final identification uses the orbit map for $p_0$, i.e., the inverse of $h \mapsto p_0 h$. Then $\bar{A}$ is submersive; $(\phi^H)'$ is given by an integral over the domain of $\bar{A}$, whereas $\phi^H$ is given by an integral over its range:

$$(\phi^H)' = \int_{\mathcal{R}\backslash A^{-1}(O)} \omega_1 \text{ and } \phi^H = \int_H \omega_2. \tag{11.6.4}$$

Note that the forms $\omega_1, \omega_2$ are not quite algebraic because they involve the $f_{ij}$ that are simply smooth functions. It is more convenient to relate them in a more algebraic version;

**Lemma 11.6.1.** *Define*

$$\tilde{\Omega} = \nu_{\mathbb{P}}^{-4} \prod_{ij \in \mathbf{E}} dz_{ij} \cdot \prod_{ij \in \mathbf{O}} \chi_{ij-}\big((z_{ij} \wedge z_{ik})(z_{il} \wedge z_{ij})(z_{ik} \wedge z_{il})^{-1}\big),$$

*which we consider as a differential* 6*-form on* $(\mathbb{P}^1)^{\mathbf{E}}$*, where*

(1) *In the product over* $ij \in \mathbf{V}$*, we always require* $\{i,j,k,l\} = \mathbf{V}$*, and the ordering of* $k$ *and* $l$ *is* opposite *for any pair of* $\chi_{ij}$ *and* $\chi_{ji}$*;*
(2) *we choose for* $ij \in \mathbf{E}$ *coordinates* $(x_{ij}, y_{ij})$ *in* $(\mathbb{A}^2)^{\mathbf{E}}$ *with* $z_{ij}$ *the corresponding points in* $(\mathbb{P}^1)^{\mathbf{E}}$*, and write*

$$z_{ij} \wedge z_{ik} := \begin{vmatrix} x_{ij} & y_{ij} \\ x_{ik} & y_{ik} \end{vmatrix}, \quad dz_{ij} = x_{ij} dy_{ij} - y_{ij} dx_{ij}.$$

*Then we have an equality of differential (regular)* 18*-forms on* $A^{-1}(O)$*:*

$$\tilde{\omega}_1^{\mathrm{alg}} = A^*(\omega_2^{\mathrm{alg}}) \wedge \tilde{\Omega},$$

*where* $\tilde{\omega}_1^{\mathrm{alg}}$ *is the numerator of* (11.6.1)*, but replacing the* $f_{ij}$ *by an* algebraic *function* $f_{ij}^{\mathrm{alg}}$ *defined on a Zariski-open subset, with the same degree of homogeneity, and making the same substitution in the definition of* $\omega_2^{\mathrm{alg}}$*.*

**11.7. Conclusion of the proof, assuming Lemma 11.6.1.** From the lemma we readily deduce that

$$\int_{\mathcal{R}\backslash A^{-1}(O)} \omega_1 = \int_{O \simeq H} \omega_2 \cdot \left( \int_{\bar{A}^{-1}(p)} \tilde{\Omega} dg \right),$$

for any choice whatsoever of $f_{ij}$. (Just integrate both sides of the lemma, but after multiplying by the ratio $f_{ij}/|f_{ij}^{\mathrm{alg}}|$.) Taking $f_{ij}$ to be supported in a very small neighbourhood of $p$, and using (11.6.4), we deduce that

$$(\phi^H)' = (\phi^H) \times \left( \int_{\bar{A}^{-1}(p_0)} \Omega(p_0) \right).$$

If we combine this with Lemma 11.6.1, we find that $(\phi^H)'$ equals $\phi^H$ multiplied by the normalization factor:

$$\nu_{\mathbb{P}}^{-4} \int_{\mathcal{R}\backslash(\mathbb{P}^1)^{\mathbf{E}}} \prod_{ij\in\mathbf{O}} \chi_{ij-}\left((z_{ij}\wedge z_{ik})(z_{il}\wedge z_{ij})(z_{ik}\wedge z_{il})^{-1}\right) \frac{\prod_{ij\in\mathbf{E}} dz_{ij}}{dg},$$

with the ordering convention for $k$ and $l$ as stated in Lemma 11.6.1. Combining this with the known relationships (11.5.1) and (11.5.3) between $\Lambda^H, \Lambda^D$ and $\phi^H, \phi^D$ completes the proof of Proposition 6.3.1, hence also Theorem 7.2.1.

**11.8. The proof of Lemma 11.6.1.** Working Zariski locally on $\mathbb{P}^1$ we fix algebraic choices of homogeneous coordinate representatives $z \mapsto \tilde{z}$ and similarly working locally on $\mathrm{PGL}_2$ we choose representatives $g \mapsto \tilde{g}$ in $\mathrm{GL}_2$. Define functions $\lambda_{ij}$ Zariski-locally on $(\mathbb{P}^1)^{\mathbf{E}} \times H$ by requiring

$$\lambda_{ij} z_{ij} \tilde{g}_i = \tilde{p}_{ij}, \quad \lambda_{ji} z_{ij} \tilde{g}_j = \tilde{p}_{ji}. \tag{11.8.1}$$

In view of (11.6.1) and (11.6.3) what we must prove is

$$\prod_{ij\in\mathbf{E}} \chi_{ij-}^{-1}(\tilde{g}_i\lambda_{ij}) = \prod_{ij\in\mathbf{O}} \chi_{ij-}\left((z_{ij}\wedge z_{ik})(z_{il}\wedge z_{ij})(z_{ik}\wedge z_{il})^{-1}\right)$$

Writing

$$\tilde{g}_i^{-1} = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}.$$

the condition (11.8.1) then amounts to

$$\begin{bmatrix} & \lambda_{12}(x_{12}, y_{12}) & \lambda_{13}(x_{13}, y_{13}) & \lambda_{14}(x_{14}, y_{14}) \\ \lambda_{21}(x_{12}, y_{12}) & & \lambda_{23}(x_{23}, y_{23}) & \lambda_{24}(x_{24}, y_{24}) \\ \lambda_{31}(x_{13}, y_{13}) & \lambda_{32}(x_{23}, y_{23}) & & \lambda_{34}(x_{34}, y_{34}) \\ \lambda_{41}(x_{14}, y_{14}) & \lambda_{42}(x_{24}, y_{24}) & \lambda_{43}(x_{34}, y_{34}) & \end{bmatrix}$$
$$= \begin{bmatrix} & (c_1, d_1) & (a_1+c_1, b_1+d_1) & (a_1, b_1) \\ (c_2, d_2) & & (a_2, b_2) & (a_2+c_2, b_2+d_2) \\ (a_3+c_3, b_3+d_3) & (a_3, b_3) & & (c_3, d_3) \\ (a_4, b_4) & (a_4+c_4, b_4+d_4) & (c_4, d_4) & \end{bmatrix}.$$

Observe that

$$\det \tilde{g}_1^{-1} = -\lambda_{12}\lambda_{13} \begin{vmatrix} x_{12} & y_{12} \\ x_{13} & y_{13} \end{vmatrix} = -\lambda_{13}\lambda_{14} \begin{vmatrix} x_{13} & y_{13} \\ x_{14} & y_{14} \end{vmatrix} = \lambda_{14}\lambda_{12} \begin{vmatrix} x_{14} & y_{14} \\ x_{12} & y_{12} \end{vmatrix}$$
$$= \lambda_{12}^2 \frac{\begin{vmatrix} x_{12} & y_{12} \\ x_{13} & y_{13} \end{vmatrix} \begin{vmatrix} x_{14} & y_{14} \\ x_{12} & y_{12} \end{vmatrix}}{\begin{vmatrix} x_{13} & y_{13} \\ x_{14} & y_{14} \end{vmatrix}},$$

and so we have

$$\psi_{12}(\tilde{g}_1)\chi_{12}^{-2}(\lambda_{12})|\lambda_{12}| = \psi_{12}^{-1}\left(\begin{bmatrix} x_{12} & y_{12} \\ x_{13} & y_{13} \end{bmatrix}\begin{bmatrix} x_{14} & y_{14} \\ x_{12} & y_{12} \end{bmatrix}\begin{bmatrix} x_{13} & y_{13} \\ x_{14} & y_{14} \end{bmatrix}^{-1}\right).$$

Similarly, for any $i \neq j$, if we let $k, l$ be the remaining two vertices, and let $\epsilon_{ijkl}$ be the sign of the permutation that sends $(\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4})$ to $(i, j, k, l)$, then we have

$$\psi_{ij}(\tilde{g}_i)\chi_{ij}^{-2}(\lambda_{ij})|\lambda_{ij}| = \chi_{ij}(\epsilon_{ijkl})\psi_{ij}^{-1}\left(\begin{bmatrix} x_{ij} & y_{ij} \\ x_{ik} & y_{ik} \end{bmatrix}\begin{bmatrix} x_{il} & y_{il} \\ x_{ij} & y_{ij} \end{bmatrix}\begin{bmatrix} x_{ik} & y_{ik} \\ x_{il} & y_{il} \end{bmatrix}^{-1}\right),$$

where for any $i > j$ we let $x_{ij} = x_{ji}$, and so on. This expression is independent of the ordering of $k$ and $l$, as expected. Note that we always have $\chi_{ij}(\epsilon_{ijkl})^2 = 1$, and so $\chi_{ij}(\epsilon_{ijkl})\chi_{ji}(\epsilon_{jilk}) = 1$. This concludes the proof.

## 12. Weyl symmetry: proof of Theorem 5.1.1

We will prove Theorem 5.1.1, namely, that the tetrahedral symbol for principal series has a $W(D_6)$-symmetry, up to an explicit cocycle.

**12.1. A Fourier duality.** We continue with the notation of § 7.1 but now specialize to the case $n = 2k$, i.e. $X$ is one-half the dimension of its ambient space. Let $X^\perp$ be the orthogonal complement to $X$ inside $F^n$ with respect to the usual pairing. We choose Haar measures $dx$ on $X$ and $dy$ on $X^\perp$ so that $dxdy = (d\mu)^n$, and $dx$ and $dy$ are Fourier transforms of each other. Recall from (2.2.5) that $\check{\chi} = \left(\chi(-1)\gamma(1,\chi)\chi_1\right)^{-1}$ is the Fourier transform of any quasi-character $\chi$ with respect to $\Psi$ and $d\mu$.

**Proposition 12.1.1.** *Suppose $n = 2k$, all the characters $\chi_i$ have the form $\alpha_i^{-1}|\cdot|^{-\frac{1}{2}}$ with the $\alpha_i$ unitary, and the integrals of $|x|^{-\frac{1}{2}} := \prod_{i=1}^n |x_i|^{-\frac{1}{2}}$ (where $x_i$ are the standard coordinates on $F^n$) over both $\mathbb{P}X$ and $\mathbb{P}X^\perp$ are convergent. Then we have the following* Fourier duality

$$\int_{\mathbb{P}X} \underline{\chi} = \prod_{i=1}^n \chi_i(-1) \cdot \int_{\mathbb{P}X^\perp} \underline{\check{\chi}}.$$

*where $\int_{\mathbb{P}X^\perp} \underline{\check{\chi}}$ is, as in* (7.1.1)*, the integral of $\prod_i \check{\chi}_i$ over the projectivation of $X^\perp$.*

*Proof.* Let $\Phi$ be a Schwartz function on $F^n$. Let $T$ be the torus $(F^\times)^n$ which we understand to act on $F^n$ in the obvious way. We equip $T$ with the Haar measure $\prod_{i=1}^n d\mu(t_i)/|t_i|$ and for $t \in T$ we write

$$|t| = \prod_{i=1}^n |t_i|, \quad \alpha(t) = \prod_{i=1}^n \alpha_i(t_i).$$

We extend them by zero to functions on $F^n$.

Let $T' \subset T$ be any complement to the central scaling copy of $F^\times$, e.g. we can take $T'$ to be the copy of $(F^\times)^{n-1}$ which scales the first $n-1$ coordintaes. The Haar measure on $T'$ is such that its product with $d\mu/|\cdot|$ on the central copy of $F^\times$ is the Haar measure on $T$ above.

We first prove that we have an equality of *absolutely convergent* integrals:

$$J_X(\alpha,\Phi) := \int_{T'\times X} \alpha(t)|t|^{\frac{1}{2}}\Phi(tx) = \int_{\mathbb{P}X} \alpha^{-1}(x)|x|^{-\frac{1}{2}} \cdot \int_{F^n} \alpha(\nu)|\nu|^{-\frac{1}{2}}\Phi(\nu),$$

with the measures specified above.

We first note that the two integrals on the right are absolutely convergent, the first by assumption, and the second because it is bounded by a product of one-variable integrals of the general form $\int_\Omega |a|^{-\frac{1}{2}} da$ for a compact set $\Omega$. To verify that the integral on the left is absolutely convergent, it is sufficient to verify that, for positive $\Phi$, the iterated integral

$$\int_X \int_{T'} |t|^{\frac{1}{2}}\Phi(tx) < \infty.$$

We can rewrite this as (iterated integrals)

$$\int_{\mathbb{P}X}\int_{\lambda\in F^\times}\int_{T'} |t|^{\frac{1}{2}}|\lambda|^{\frac{n}{2}}\Phi(\lambda t x) = \int_{\mathbb{P}X}\int_T |t|^{\frac{1}{2}}\Phi(tx).$$

The inner integral is, by what we already noted, finite as long as $x$ lies on no coordinate hyperplane, and its value is equal to a constant multiple of $|x|^{-\frac{1}{2}}$; and by assumption $\int_{\mathbb{P}X}|x|^{-\frac{1}{2}} < \infty$. So the left hand integral, too, is absolutely convergent.

Since the integral $J(\alpha,\Phi)$ is absolutely convergent, it can be evaluted in whatever order we please, and we may now compute:

$$\begin{aligned}
J_X(\alpha,\Phi) &= \int_{T'} \alpha(t)|t|^{\frac{1}{2}} \int_X \Phi(tx)dx \\
&= \int_{T'} \alpha(t)|t|^{-\frac{1}{2}} \int_{X^\perp} \check{\Phi}(t^{-1}y)dy \\
&= \int_{T'} \alpha^{-1}(t)|t|^{\frac{1}{2}} \int_{X^\perp} \check{\Phi}(ty)dy \\
&= J_{X^\perp}(\alpha^{-1},\check{\Phi}),
\end{aligned}$$

where, at the third step, we inverted $t$, and at the first stage, we used Fourier duality on $F^n$ for the Schwartz function $\Phi$ and the distribution $dx/|t|$:

$$\int_X \Phi(tx)dx = \int_X \Phi(x)\cdot\frac{dx}{|t|} = \int_{X^\perp}\check{\Phi}(y)\cdot|t|dy = \int_{X^\perp}\check{\Phi}(t^{-1}y)dy.$$

That is to say, we have proved that

$$\int_{\mathbb{P}X} \alpha^{-1}(x)|x|^{-\frac{1}{2}} \cdot \int_{F^n} \alpha(\nu)|\nu|^{-\frac{1}{2}}\Phi(\nu)$$

is symmetric under replacement of $X$ by $X^\perp$, $\Phi$ by $\check{\Phi}$ and $\alpha$ by $\alpha^{-1}$. Combined with another Fourier duality from § 2.3:

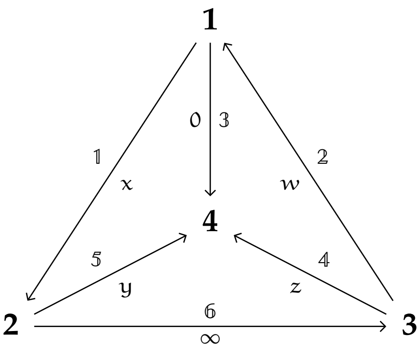
$$\int_{F^n} \alpha(\nu)|\nu|^{-\frac{1}{2}}\Phi(\nu) = \int_{F^n} \mathcal{F}(\alpha|\cdot|^{-\frac{1}{2}})(-\nu)\check{\Phi}(\nu) \overset{(2.2.5)}{=} \gamma(1,\alpha|\cdot|^{-\frac{1}{2}})^{-1}\int_{F^n}\alpha^{-1}(\nu)|\nu|^{-\frac{1}{2}}\check{\Phi}(\nu),$$

we have

$$\gamma(1, \alpha|\cdot|^{-\frac{1}{2}})^{-1} \int_{\mathbb{P}X} \alpha^{-1}(x)|x|^{-\frac{1}{2}} = \prod_i \alpha_i(-1) \int_{\mathbb{P}X^\perp} \alpha(x)|x|^{-\frac{1}{2}},$$

which translates to the desired equality. ∎

**12.2. The Weyl symmetry.** In order to explain the indexing of characters here, we combine (4.3.1) and (7.2.3) in the following diagram:




where we label each edge with both a blackboard bold number and a coordinate that is consistent with Theorem 7.2.1, which we proved in § 11.

We express $\{\Pi\}$ in terms of the hypergeometric integral (7.2.1). Let us first recall some abbreviations that we will use. We shall use the following type of abbreviation (given by an example; cf. § 4.4):

$$\chi_{\mathbf{2\bar{3}\bar{4}}} \coloneqq \chi_{\mathbf{12}}\chi_{\mathbf{13}}^{-1}\chi_{\mathbf{14}}^{-1}, \quad \gamma_{\mathbf{2\bar{3}\bar{4}}} \coloneqq \gamma\Big(\frac{1}{2}, \chi_{\mathbf{12}}\chi_{\mathbf{13}}^{-1}\chi_{\mathbf{14}}^{-1}\Big).$$

We further abridge the special cases when there are one or two inverted characters:

$$\chi_i^l \coloneqq \chi_{jk\bar{l}} = \frac{\chi_{ij}\chi_{ik}}{\chi_{il}} \quad \text{and} \quad \tilde{\chi}_i^l \coloneqq \chi_{\bar{j}\bar{k}l} = \frac{\chi_{il}}{\chi_{ij}\chi_{ik}},$$

and by extension

$$\gamma_i^l \coloneqq \gamma_{jk\bar{l}} = \gamma\Big(\frac{1}{2}, \chi_i^l\Big).$$

The convergence claim of Proposition 6.3.1 (proved in § 10.2) then allows us to apply Proposition 12.1.1 to (7.2.1), and we arrive at

$$\begin{aligned}\{\Pi\}\sqrt{\gamma\Big(\frac{1}{2}, S^-\Big)} &= \nu_{\mathbb{P}}^{-1}\left[\gamma_{\mathbf{2}}^{\mathbf{3}}\gamma_{\mathbf{4}}^{\mathbf{1}}\gamma_{\mathbf{3}}^{\mathbf{2}}\gamma_{\mathbf{1}}^{\mathbf{4}}\gamma_{\mathbf{1}}^{\mathbf{3}}\gamma_{\mathbf{4}}^{\mathbf{3}}\gamma_{\mathbf{4}}^{\mathbf{2}}\gamma_{\mathbf{1}}^{\mathbf{2}}\right]^{-1} \\ &\times \int_{[x,y,z,w]\in\mathbb{P}^3(F)} \tilde{\chi}_{\mathbf{2}-}^{\mathbf{3}}(x)\tilde{\chi}_{\mathbf{4}-}^{\mathbf{1}}(y)\tilde{\chi}_{\mathbf{3}-}^{\mathbf{2}}(z)\tilde{\chi}_{\mathbf{1}-}^{\mathbf{4}}(w)\tilde{\chi}_{\mathbf{1}-}^{\mathbf{3}}(w-x)\tilde{\chi}_{\mathbf{4}-}^{\mathbf{3}}(x-y)\tilde{\chi}_{\mathbf{4}-}^{\mathbf{2}}(y-z)\tilde{\chi}_{\mathbf{1}-}^{\mathbf{2}}(z-w),\end{aligned} \tag{12.2.1}$$

where we used (2.2.5)

$$\check{\chi} = \big(\chi(-1)\gamma(1,\chi)\chi_1\big)^{-1},$$

and that

$$(\chi_2^3\chi_4^1\chi_3^2\chi_1^4)(-1) = (\chi_1^3\chi_4^3\chi_4^2\chi_1^2)(-1) = 1.$$

The new integral in (12.2.1) may be interpreted as the integral (7.2.1) with a different set of $\chi_{ij}$, the latter obtained by performing what we will call an *inv*-Regge symmetry through the pair of opposite edges $\mathbb{2}$ and $\mathbb{5}$. (for discussion, see § 4.3; it relates to the usual Regge symmetry by composition with inversion of each character). The said transformation may formally be written as follows:

$$\begin{aligned} \mathbb{1} &\mapsto \mathbb{1}' := \frac{\mathbb{1}+\bar{\mathbb{3}}+\bar{\mathbb{4}}+\bar{\mathbb{6}}}{2}, & \mathbb{2} &\mapsto \mathbb{2}' := \bar{\mathbb{2}}, \\ \mathbb{3} &\mapsto \mathbb{3}' := \frac{\bar{\mathbb{1}}+\mathbb{3}+\bar{\mathbb{4}}+\bar{\mathbb{6}}}{2}, & \mathbb{4} &\mapsto \mathbb{4}' := \frac{\bar{\mathbb{1}}+\bar{\mathbb{3}}+\mathbb{4}+\bar{\mathbb{6}}}{2}, \\ \mathbb{5} &\mapsto \mathbb{5}' := \bar{\mathbb{5}}, & \mathbb{6} &\mapsto \mathbb{6}' := \frac{\bar{\mathbb{1}}+\bar{\mathbb{3}}+\bar{\mathbb{4}}+\mathbb{6}}{2}, \end{aligned}$$

where $\bar{\mathbb{1}}$ means formally negating $\mathbb{1}$ and so on. Thus, for example, the new character $\chi_{\mathbb{1}'}$ associated to *oriented* edge **12** after the transformation satisfies the relation

$$\chi_{\mathbb{1}'}^2 = \chi_{\mathbb{1}}\chi_{\mathbb{3}}^{-1}\chi_{\mathbb{4}}^{-1}\chi_{\mathbb{6}}^{-1} = \chi_{\mathbf{12}}\chi_{\mathbf{41}}\chi_{\mathbf{43}}\chi_{\mathbf{32}},$$

and similarly for other edges. This does not uniquely determine $\chi_{\mathbb{1}'}$ and so on: indeed, the character appearing on the right need not even have a square root. However, this is not an issue: it is simply a reflection of the fact that the map (5.1.4) is not surjective; rather, this transformation does determine the (new) integrand in (7.2.1). For example, the first character $\chi_{2-}^3$ in the integrand is the same as

$$\chi_{\bar{\mathbb{1}}\mathbb{5}\bar{\mathbb{6}}-} := \chi_{\mathbb{1}}^{-1}\chi_{\mathbb{5}}\chi_{\mathbb{6}}^{-1}|\cdot|^{-\frac{1}{2}},$$

and after the inv-Regge symmetry, we have

$$\chi_{\bar{\mathbb{1}}'\mathbb{5}'\bar{\mathbb{6}}'-} = \chi_{\mathbb{3}}\chi_{\mathbb{4}}\chi_{\mathbb{5}}^{-1}|\cdot|^{-\frac{1}{2}} = \chi_{\mathbb{3}\mathbb{4}\bar{\mathbb{5}}-} = \tilde{\chi}_{4-}^{2}.$$

Thus, the new edge integral after inv-Regge symmetry is the integral in the expression:

$$I^{E}(\mathbb{P}_F^1,\psi') = \nu_{\mathbb{P}}\int \tilde{\chi}_{4-}^2(x-y)\tilde{\chi}_{4-}^3(y-z)\tilde{\chi}_{1-}^3(z-w)\tilde{\chi}_{1-}^2(w-x)\tilde{\chi}_{3-}^2(x)\tilde{\chi}_{4-}^1(y)\tilde{\chi}_{2-}^3(z)\tilde{\chi}_{1-}^4(w)$$

which is easily seen equal to the integral in (12.2.1) by exchanging $x$ with $z$ (and using the fact that $\tilde{\chi}_4^2\tilde{\chi}_4^3\tilde{\chi}_1^3\tilde{\chi}_1^2(-1) = 1$).

Let $\{\Pi'\}$ be the tetrahedral symbol associated with the new set of characters after performing the said inv-Regge symmetry, and $\gamma(\frac{1}{2},(S^-)')$ be the corresponding $\gamma$-factor (see § 5.1). Then we have by (6.3.1)

$$\{\Pi'\}\sqrt{\gamma\Big(\frac{1}{2},(S^-)'\Big)} = \nu_{\mathbb{P}}^{-2}\, I^{E}(\mathbb{P}_F^1,\psi').$$

Therefore, we showed that

$$\{\Pi\} = \{\Pi'\} \times \frac{\left[\gamma_2^3\gamma_4^1\gamma_3^2\gamma_1^4\gamma_1^3\gamma_4^3\gamma_4^2\gamma_1^2\right]^{-1}\sqrt{\gamma(\frac{1}{2},(S^-)')}}{\sqrt{\gamma(\frac{1}{2},S^-)}}. \tag{12.2.2}$$

**Proposition 12.2.1.** *We have an equality up to sign:*

$$\{\Pi\} = \{\Pi'\}.$$

*Proof.* It amounts to showing that the fraction on the right-hand side of (12.2.2) equals $\pm 1$. Indeed, by definition,

$$\begin{aligned}\gamma\Big(\frac{1}{2}, S^-\Big) &= \gamma_{2\bar3\bar4}\gamma_{\bar2 3\bar4}\gamma_{\bar2\bar3 4}\gamma_{\bar2\bar3\bar4} \times \gamma_{1\bar3\bar4}\gamma_{\bar1 3\bar4}\gamma_{\bar1\bar3 4}\gamma_{\bar1\bar3\bar4} \\ &\quad \times \gamma_{1\bar2\bar4}\gamma_{\bar1 2\bar4}\gamma_{\bar1\bar2 4}\gamma_{\bar1\bar2\bar4} \times \gamma_{1\bar2\bar3}\gamma_{\bar1 2\bar3}\gamma_{\bar1\bar2 3}\gamma_{\bar1\bar2\bar3},\end{aligned}$$

whereas after the inv-Regge symmetry we find

$$\begin{aligned}\gamma\Big(\frac{1}{2}, (S^-)'\Big) &= \gamma_{23\bar4}\gamma_{1\bar2 4}\gamma_{\bar2 34}\gamma_{\bar1\bar2 4} \times \gamma_{\bar1\bar2\bar3}\gamma_{1\bar2 3}\gamma_{\bar1\bar3\bar4}\gamma_{\bar1\bar3 4} \\ &\quad \times \gamma_{\bar1\bar2\bar4}\gamma_{2\bar3 4}\gamma_{\bar2\bar3\bar4}\gamma_{1\bar2\bar4} \times \gamma_{12\bar3}\gamma_{1\bar3 4}\gamma_{\bar1 23}\gamma_{1\bar3\bar4}.\end{aligned}$$

As a result, we have (up to sign)

$$\begin{aligned}\sqrt{\frac{\gamma(\frac{1}{2},(S^-)')}{\gamma(\frac{1}{2},S^-)}} &= \sqrt{\frac{\gamma_{23\bar4}}{\gamma_{\bar2\bar3 4}}\frac{\gamma_{2\bar3 4}}{\gamma_{\bar2 3\bar4}}\frac{\gamma_{\bar2 34}}{\gamma_{2\bar3\bar4}}\frac{\gamma_{1\bar3 4}}{\gamma_{\bar1 3\bar4}}\frac{\gamma_{1\bar2 4}}{\gamma_{\bar1 2\bar4}}\frac{\gamma_{\bar1 23}}{\gamma_{1\bar2\bar3}}\frac{\gamma_{1\bar2 3}}{\gamma_{\bar1 2\bar3}}\frac{\gamma_{12\bar3}}{\gamma_{\bar1\bar2 3}}} \\ &= \gamma_{23\bar4}\gamma_{2\bar3 4}\gamma_{\bar2 34}\gamma_{1\bar3 4}\gamma_{1\bar2 4}\gamma_{\bar1 23}\gamma_{1\bar2 3}\gamma_{12\bar3} \\ &= \gamma_1^4\gamma_1^3\gamma_1^2\gamma_2^3\gamma_3^2\gamma_4^1\gamma_4^2\gamma_4^3,\end{aligned}$$

where we again used the symmetry properties (2.2.6) of the $\gamma$-factor; and the fact that

$$\big(\chi_2^3\chi_4^1\chi_3^2\chi_1^4\chi_1^3\chi_4^3\chi_4^2\chi_1^2\big)(-1) = 1.$$

This finishes the proof. ∎

## 12.3. The proof of $W(D_6)$-symmetry, completed.

*Proof of Theorem 5.1.1.* Let us begin with the first claim (1), that $\{\chi\}^2$ descends to $D_6 \otimes \widehat{F^\times}$ and is $W(D_6)$-invariant. Formula (7.2.2) shows that $\{\chi\}^2$ depends only on characters of the form $\chi_{ij}^{\pm}\chi_{ik}^{\pm}\chi_{il}^{\pm}$, which gives the descent. In the language of § 4.3, $\{\chi\}^2$ is evidently invariant both by the group of orientation reversals (since these do not change the isomorphism class of the underlying representation) and also by the group of tetrahedral symmetries (by the way we defined it). Finally, Proposition 12.2.1 shows that $\{\chi\}^2$ is invariant by at least one Regge symmetry; together with the prior groups, this generates all of $W(D_6)$. This concludes the proof of the first claim.

We now pass to claim (2). By (6.3.1) we may take $I(\chi)$ to be the integral $I^E(\mathbb{P}^1_F, \psi)$, multiplied by $\nu_{\mathbb{P}}^{-2}$. Then $a(w, \psi) = I(w^{-1}\chi)/I(\chi)$ is automatically a cocycle of $W(D_6)$

valued in functions of $\psi$. By what we have already proved, namely, that $\{\chi\}^2$ is $W(\mathsf{D}_6)$-invariant, the validity of (5.1.5) for *some* choice of signs $\iota(w,\chi)$ follows; it remains to describe the signs.

Since $\mathfrak{a}$ is a cocycle, it suffices to compute $\iota$ on the generators of $W$. By Lemma 4.3.3, it suffices to consider $\mathfrak{I}$, $\mathfrak{T}$ and a single element in $\mathfrak{R}$. Some of them are easy: the **V**-tetrahedral group $\mathfrak{T}$ preserves $S^+$ and $S^-$, and evidently also $I^{\mathbf{E}}(\mathbb{P}^1_F,\psi)$, so they have trivial signs; by (12.2.1), the one particular inv-Regge symmetry in Proposition 12.2.1 also has trivial sign.

We then compute the sign for $w$ being the edge flipping operations, and it suffices to assume that the edge is **12**, and that all $\chi_{ij}$ are unitary. We will need to use the results from §§ 14.3 and 14.6 (whose argument is purely analytic based on the integral formula (6.3.1) and independent of Theorem 5.1.1). In short, (6.3.1) may be written as $f(1)$ for some smooth function $f$ on $F$ whose Mellin transform (as a function on $\widehat{F^\times}$) is

$$\lambda \longmapsto \left[\gamma(\chi^4_{1+})\gamma(\chi^3_{2+})\gamma(\chi^1_{4+})\gamma(\chi^2_{3+})\right]^{-1}\frac{\gamma(\lambda\otimes A_+)}{\gamma(\lambda\otimes B)},$$

where

$$\begin{aligned} A &= \{\chi_{12}\chi_{31}\chi_{41}, \chi_{21}\chi_{41}\chi_{31}, \chi_{43}\chi_{31}\chi_{23}, \chi_{34}\chi_{31}\chi_{23}\} \\ B &= \left\{\chi^2_{31}, \chi_{42}\chi_{41}\chi_{31}\chi_{23}, \chi_{41}\chi_{31}\chi_{24}\chi_{23}, 1\right\}. \end{aligned}$$

It is clear that exchanging **12** with **21** only changes the constant factor $\left[\gamma(\chi^4_{1+})\gamma(\chi^3_{2+})\right]^{-1}$ in the Mellin transform into $\left[\gamma\big((\chi^3_{1-})^{-1}\big)\gamma\big((\chi^4_{2-})^{-1}\big)\right]^{-1}$. Therefore, we have

$$\begin{aligned} \frac{I(w^{-1}\chi)}{I(\chi)} &= \frac{\gamma(\chi^4_{1+})\gamma(\chi^3_{2+})}{\gamma\big((\chi^3_{1-})^{-1}\big)\gamma\big((\chi^4_{2-})^{-1}\big)} \\ &= \chi^3_1(-1)\chi^4_2(-1)\gamma(\chi^4_{1+})\gamma(\chi^3_{2+})\gamma(\chi^3_{1+})\gamma(\chi^4_{2+}) \\ &= \big(\chi_{13}\chi_{14}\chi_{23}\chi_{24}\big)(-1)\gamma(\chi^4_{1+})\gamma(\chi^3_{2+})\gamma(\chi^3_{1+})\gamma(\chi^4_{2+}). \end{aligned} \tag{12.3.1}$$

This shows that the sign for flipping the edge **12** is $\big(\chi_{13}\chi_{14}\chi_{23}\chi_{24}\big)(-1)$.

Let $s_{ij}$ be the element flipping edge $ij$. We show here that

$$\iota(s_{ij}s_{ik}s_{il},\chi) = \iota(s_{ij}s_{jk}s_{ki},\chi) = \chi_{ij}\chi_{ik}\chi_{il}(-1) = \chi_{jkl}(-1), \tag{12.3.2}$$

where $\{i,j,k,l\} = \mathbf{V}$. Then (12.3.1) implies that

$$\begin{aligned} \frac{I(s_{ik}^{-1}s_{ij}^{-1}\chi)}{I(s_{ij}^{-1}\chi)} &= \big(\chi_{ij}^{-1}\chi_{il}\chi_{kj}\chi_{kl}\big)(-1)\gamma\Big(\frac{1}{2},\frac{\chi_{ik}\chi_{il}}{\chi_{ij}^{-1}}\Big)\gamma\Big(\frac{1}{2},\frac{\chi_{ik}\chi_{ij}^{-1}}{\chi_{il}}\Big)\gamma\Big(\frac{1}{2},\frac{\chi_{ki}\chi_{kl}}{\chi_{kj}}\Big)\gamma\Big(\frac{1}{2},\frac{\chi_{ki}\chi_{kj}}{\chi_{kl}}\Big) \\ &= \big(\chi_{ij}\chi_{il}\chi_{kj}\chi_{kl}\big)(-1)\gamma(\chi_{jkl+})\gamma(\chi_{\bar{j}k\bar{l}+})\gamma(\chi_{i\bar{j}l+})\gamma(\chi_{ij\bar{l}+}) \end{aligned}$$

And one can compute other quotients $I(s_{il}^{-1}s_{ik}^{-1}s_{ij}^{-1}\chi)/I(s_{ik}^{-1}s_{ij}^{-1}\chi)$ and so on. On the other hand, the weights in $S^+\cap s_{ij}s_{ik}s_{il}(S^-)$ are precisely

$$\chi_{i\bar{j}k}, \chi_{ij\bar{k}}, \quad \chi_{i\bar{k}l}, \chi_{ik\bar{l}}, \quad \chi_{i\bar{j}l}, \chi_{ij\bar{l}}, \quad \chi_{jkl}, \chi_{\bar{j}kl}, \chi_{j\bar{k}l}, \chi_{jk\bar{l}}.$$

The claim (12.3.2) is then an easy (albeit a bit tedious) exercise.

The longest element $w_0$, which flips all edges simultaneously, can be written as the product of $s_{ij}s_{ik}s_{il}$ and $s_{jk}s_{kl}s_{lj}$. Repeating the same argument for these two elements, we see that $\iota(w_0, \chi) = 1$.

Lastly, the inv-Regge symmetry in Proposition 12.2.1 is the element $r_{\mathbb{25}}w_0 = w_0 r_{\mathbb{25}}$. Using the fact that $\iota(w_0, \chi) = 1$ and $S^+ \cap w_0(S^-) = S_+$, we then can show that $\iota(r_{\mathbb{25}}, \chi) = 1$. Conjugating using $\mathfrak{T}$, or more precisely by cyclically permuting $\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$, the same also holds for $r_{\mathbb{14}}$ and $r_{\mathbb{36}}$. This finishes the proof. ∎

## 13. Computations for unramified principal series

In this section, we prove Theorem 5.2.1 by direct computations, partially assisted by a computer. Using Proposition 6.2.1, the statement to be proved is as follows:

$$(1 - q^{-2})^5 \, I^{\mathbf{V}}(\mathcal{R}/K, \varphi) L(1, \mathrm{ad}) = \mathrm{Tr}(q^{-\frac{1}{2}}\sigma, \mathbb{C}[P]). \tag{13.0.1}$$

Our proof breaks into several steps:

(1) We first compute $I^{\mathbf{V}}(\mathcal{R}/K, \varphi)$ using the Bruhat–Tits tree of $\mathcal{R} \cong \mathrm{PGL}_2$ as a sum of terms, each of which is a product of several geometric series; see § 13.1.2.

(2) On the dual side, the representation $\mathbb{C}[P]$ of $\mathrm{Spin}_{12}$ decomposes in a very nice way by using the Cartan map. Then the trace of $q^{-\frac{1}{2}}\sigma$ can be computed using Weyl character formula, see § 13.2.4.

At this point, we should be able to prove (13.0.1) by direct comparison, since both sides are algebraic expressions. However, the length of the expressions seems too large for our computer to handle efficiently, and so we opt for a more indirect approach using analysis. Regard both sides of (13.0.1) as functions of on the six-dimensional torus $\mathsf{D}_6 \otimes \mathbb{T}$ of possible $\sigma$. We shall then verify that:

(3) Both sides of (13.0.1) have at most simple poles along the locus where some eigenvalue of $\sigma$ acting on $S$ equals $q^{\frac{1}{2}}$. See Proposition 13.1.5 and § 13.2.4.

(4) These poles have the same residues. See § 13.3 for the (computer-assisted) computation.

Therefore, the difference between the two sides defines a regular function on $\mathsf{D}_6 \otimes \mathbb{T} \simeq \mathbb{G}_m^6$. To conclude it is constant, we use the following observations:

(5) If we restrict either the left-hand side or the right-hand side to a generic one-parameter subgroup of the torus $\mathsf{D}_6 \otimes \mathbb{T}$ they remain bounded at infinity. (See § 13.1.3 and § 13.2.5).

Here "generic" means that the coordinates of the one-parameter subgroup is permitted to avoid a finite set of hyperplanes. One readily verifies that a regular function on $\mathbb{G}_m^n$ that remains bounded along a generic one-parameter subgroup is constant. So the difference between the left-hand and right-hand sides must be constant. We can conclude by showing that the desired equality holds at a single value of $\sigma$.

Let us set up notation. In the statement of Theorem 5.2.1, we have $\chi \in \mathcal{X}_0$ with image $\sigma \in \mathsf{D}_6 \otimes \mathbb{T}$; we denote by $x_{ij} = \chi_{ij}(\varpi)$ the value of $\chi_{ij}$ at the uniformizer, so that the various $x_{ij}$ for $i < j$ provide coordinates on $\tilde{D}_6 \otimes \mathbb{T}$. Let $\sigma \in D_6 \otimes \mathbb{T}$ be as in the statement of the theorem, whereas the image of $\sigma$ under the six standard coordinate functionals are

$$x_{12}x_{34}, x_{12}x_{34}^{-1}, \quad x_{13}x_{24}, x_{13}x_{24}^{-1}, \quad x_{14}x_{23}, x_{14}x_{23}^{-1},$$

as well as their inverses.

**13.1. The computation based on Bruhat–Tits tree.** Recall that the quotient set $\mathcal{R}/K$ can be naturally identified with the set of vertices of a tree, namely the Bruhat–Tits tree of $\mathcal{R}$. The $K$-orbits of those vertices, in other words, the double quotient $K\backslash\mathcal{R}/K$, can be identified with $\mathbb{N}$: the identity coset $K$ is sent to $0$, and in general we send a double coset to its distance from the orbit $K$.

With the notations in § 6.2, we note that the spherical vector $v_{ij}$ can be represented by the eigenfunction of the adjacency matrix of the tree, with eigenvalue $q^{-\frac{1}{2}}(x_{ij} + x_{ij}^{-1})$, where $x_{ij} = \chi_{ij}(\varpi)$ is the value of $\chi_{ij}$ at the uniformizer. Recall that for any $x \in \mathbb{C}$, the corresponding Hecke eigenspace is generated by a function on $\mathbb{N} \cong K\backslash\mathcal{R}/K$:

$$f_x(n) = \frac{qx - x^{-1}}{(1+q)(x - x^{-1})}(q^{-\frac{1}{2}}x)^n + \frac{x - qx^{-1}}{(1+q)(x - x^{-1})}(q^{-\frac{1}{2}}x^{-1})^n,$$

where we, for now, naively normalize $f_x$ so that $f_x(0) = 1$. Let $f_{ij} = f_{x_{ij}}$.

The space $X^{\mathbf{V}}$ is simply the moduli space of 4 labeled vertices on the Bruhat–Tits tree, and so $\mathcal{R}\backslash X^{\mathbf{V}}$ may be identified with the subspace where the vertex **1** sits at the root (that is, the trivial coset $K$). The function $\varphi_{ij}$ is simply the value of $f_{ij}$ evaluated at the distance between vertices $i$ and $j$. Note that the opposite orientation of the edge $ij$ (in other words, using $f_{ji}$ instead of $f_{ij}$) does not change $\varphi_{ij}$, because the formula above is invariant under $x \leftrightarrow x^{-1}$. For definiteness we shall choose the orientations so that $i < j$, which is consistent with other parts of this paper.

**13.1.1.** The generic configuration patterns of 4 vertices on a tree (with **1** at the root) are, as follows, put into three (not disjoint) classes **A**, **B**, **C**:

$$\begin{tikzcd}
 & \mathbf{1} \arrow[d, "a"] & & \mathbf{A} \\
 & \bullet \arrow[dl, "b"'] \arrow[dr, "c"] & & \\
\mathbf{2} & & \bullet \arrow[dl, "d"'] \arrow[dr, "e"] & \\
 & \mathbf{3} & & \mathbf{4}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{1} \arrow[d, "a"] & & \mathbf{B} \\
 & \bullet \arrow[dl, "b"'] \arrow[dr, "c"] & & \\
\mathbf{3} & & \bullet \arrow[dl, "d"'] \arrow[dr, "e"] & \\
 & \mathbf{2} & & \mathbf{4}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{1} \arrow[d, "a"] & & \mathbf{C} \\
 & \bullet \arrow[dl, "b"'] \arrow[dr, "c"] & & \\
\mathbf{4} & & \bullet \arrow[dl, "d"'] \arrow[dr, "e"] & \\
 & \mathbf{2} & & \mathbf{3}
\end{tikzcd}
\tag{13.1.1}$$

where any "line" above signifies a connecting path rather than just an edge of the tree, and the letters $a$ through $e$ are the lengths of the paths. Any of these numbers can be $0$; in particular when $c = 0$, the three classes coincide, and in this case we will use extra care.

Given any pattern as in (13.1.1), say in class **A**, with fixed $a, \dots, e$, then its contribution towards the vertex integral $I^{\mathbf{V}}(X, \varphi)$ is given by

$$f_{12}(a+b)f_{13}(a+c+d)f_{14}(a+c+e)f_{23}(b+c+d)f_{24}(b+c+e)f_{34}(d+e)$$

| # | Zero conditions | $C_{abcde}$ |
|---|---|---|
| 1 | $a=b=c=d=e=0$ | $1$ |
| 2 | $abcde \neq 0$ | $(1+q^{-1})(1-q^{-1})^2$ |
| 3 | $c=0$, $abde \neq 0$ | $(1+q^{-1})(1-q^{-1})(1-2q^{-1})$ |
| 4 | $ab=0$, $de\neq 0$, $a+b+c\neq 0$ | $(1+q^{-1})(1-q^{-1})$ |
| 5 | $ab\neq 0$, $de=0$, $c+d+e\neq 0$ | $(1+q^{-1})(1-q^{-1})$ |
| 6 | otherwise | $(1+q^{-1})$ |

TABLE 1. Values of $C_{abcde}$

multiplied by the volume $\mu$ of the subset of $\mathcal{R}\backslash X^V$ that gives this pattern. Note that we put **1** at the root, which is the same as identifying $\mathcal{R}\backslash X^V$ with the double quotient $K\backslash X^3$, where $X^3$ is indexed by $\mathbf{2},\mathbf{3},\mathbf{4}$, and $K$ acts diagonally.

Since the volume of $K$ is normalized to be 1, and since $K$ acts transitively on the set of all configurations in class **A** with the same pattern (that is, same numbers $a,\dots,e$), the volume $\mu$ is just the number of elements in this set of configurations. It is then easy to see that $\mu$ is of the order $q^{a+b+c+d+e}$, but with an additional factor $C_{abcde}$ depending on whether any of $a,\dots,e$ is $0$ or not. For example, if none of the numbers are $0$, then there are $(1+q^{-1})q^{a+b}$ different choices to place **2**, and then $(1-q^{-1})q^{c+d}$ choices to place **3**, and finally $(1-q^{-1})q^{e}$ choices to place **4**. In this case, $C_{abcde}=(1+q^{-1})(1-q^{-1})^2$. The values of $C_{abcde}$ are listed in Table 1.

The classes **B** and **C** are treated similarly, but with the complication that when $c=0$, they duplicate cases already considered for **A**. To eliminate such duplications, we let $Z_c = 1/3$ if $c=0$ and 1 otherwise, and multiply everything in all three classes by $Z_c$.

**13.1.2.** Thus, $I^V(X,\varphi)$ is equal to the series

$$\sum_{a,b,c,d,e}\Big[f_{\mathbf{12}}(a+b)f_{\mathbf{13}}(a+c+d)f_{\mathbf{14}}(a+c+e)f_{\mathbf{23}}(b+c+d)f_{\mathbf{24}}(b+c+e)f_{\mathbf{34}}(d+e)$$
$$+f_{\mathbf{13}}(a+b)f_{\mathbf{12}}(a+c+d)f_{\mathbf{14}}(a+c+e)f_{\mathbf{23}}(b+c+d)f_{\mathbf{34}}(b+c+e)f_{\mathbf{24}}(d+e)$$
$$+f_{\mathbf{14}}(a+b)f_{\mathbf{13}}(a+c+d)f_{\mathbf{12}}(a+c+e)f_{\mathbf{34}}(b+c+d)f_{\mathbf{24}}(b+c+e)f_{\mathbf{23}}(d+e)\Big]q^{a+b+c+d+e}C_{abcde}Z_c,$$

where $a,\dots,e$ range in $\mathbb{N}$.

**13.1.3.** *Boundedness of the vertex integral.* We now show that

$$(1-q^{-2})^5\, I^V(\mathcal{R}/K,\varphi)L(1,\mathrm{ad})$$

remains bounded when $\sigma$ varies through a *generic* one-parameter torus.

Explicitly, let $x_{ij}=t^{n_{ij}}$ for $n_{ij}\in\mathbb{Z}$; we will prove the boundedness as $t\to 0$ so long as all the $n_{ij}$ are nonzero. The factor $L(1,\mathrm{ad})$ is, up to constants, a product of terms $(1-q^{-1}t^{\pm 2n_{ij}})^{-1}$. Such factors approach 1 as $t\to 0$ if the exponent is positive, $1-q^{-1}$ if it is zero, and zero if it is negative. Therefore, it suffices to prove the boundedness for $I^V(\mathcal{R}/K,\varphi)$.

The constants $C_{abcde}$ and $Z_c$ are independent of $x_{ij}$ and for $abcde \neq 0$ they stay the same constants respectively. It suffices to look at the summation

$$\sum_{a,b,c,d,e} f_{12}(a+b)f_{13}(a+c+d)f_{14}(a+c+e) \\ \times f_{23}(b+c+d)f_{23}(b+c+e)f_{34}(d+e)q^{a+b+c+d+e},$$

because the other two summands are similar. Each $f_{ij}(n)$ is the sum of two terms:

$$\frac{qx_{ij}-x_{ij}^{-1}}{(1+q)(x_{ij}-x_{ij}^{-1})}(q^{-\frac{1}{2}}x_{ij})^n, \quad \frac{x_{ij}-qx_{ij}^{-1}}{(1+q)(x_{ij}-x_{ij}^{-1})}(q^{-\frac{1}{2}}x_{ij}^{-1})^n,$$

where the coefficients in front of $(q^{-\frac{1}{2}}x_{ij})^n$ and $(q^{-\frac{1}{2}}x_{ij}^{-1})^n$ are uniformly bounded when $x_{ij}$ goes to 0 or $\infty$. Expanding the products of $f_{ij}$s in the summation, we see that we only need to bound the series

$$\begin{aligned}
&\sum_{a,b,c,d,e} (q^{-\frac{1}{2}}t^{n_{12}})^{a+b}(q^{-\frac{1}{2}}t^{n_{13}})^{a+c+d}(q^{-\frac{1}{2}}t^{n_{14}})^{a+c+e} \\
&\qquad \times (q^{-\frac{1}{2}}t^{n_{23}})^{b+c+d}(q^{-\frac{1}{2}}t^{n_{24}})^{b+c+e}(q^{-\frac{1}{2}}t^{n_{34}})^{d+e}q^{a+b+c+d+e} \\
=&\sum_{a,b,c,d,e} \left(q^{-\frac{a}{2}}t^{a(n_{12}+n_{13}+n_{14})}\right)\left(q^{-\frac{b}{2}}t^{b(n_{12}+n_{23}+n_{24})}\right)\left(q^{-c}t^{c(n_{13}+n_{14}+n_{23}+n_{24})}\right) \\
&\qquad \times \left(q^{-\frac{d}{2}}t^{d(n_{13}+n_{23}+n_{34})}\right)\left(q^{-\frac{e}{2}}t^{e(n_{14}+n_{24}+n_{34})}\right) \\
&= \frac{1}{1-q^{-\frac{1}{2}}t^{n_{12}+n_{13}+n_{14}}} \times \frac{1}{1-q^{-\frac{1}{2}}t^{n_{12}+n_{23}+n_{24}}} \times \frac{1}{1-q^{-1}t^{n_{13}+n_{14}+n_{23}+n_{24}}} \\
&\qquad \times \frac{1}{1-q^{-\frac{1}{2}}t^{n_{13}+n_{23}+n_{34}}} \times \frac{1}{1-q^{-\frac{1}{2}}t^{n_{14}+n_{24}+n_{34}}}. \qquad (13.1.2)
\end{aligned}$$

Clearly when $t \to 0$ or $t \to \infty$, the above series is a product of 1, 0, $(1-q^{-\frac{1}{2}})^{-1}$ or $(1-q^{-1})^{-1}$, hence bounded.

**13.1.4.** *Poles of the vertex integral.*

**Proposition 13.1.5.** *The expression* $I^{\mathbf{V}}(\mathcal{R}/K, \varphi)L(1, \mathrm{ad})$ *viewed as a rational function in variables* $x_{ij}$ *and* $q^{-\frac{1}{2}}$, *has poles at hypersurfaces*

$$1-q^{-\frac{1}{2}}x_{ij}^{\pm}x_{ik}^{\pm}x_{il}^{\pm},$$

*where* $\mathbf{V} = \{i, j, k, l\}$, *and nowhere else, that is to say, only poles at points where* $\sigma$ *has an eigenvalue* $q^{\frac{1}{2}}$ *in the half-spin representation* $S^{ev} = S$.

*Proof.* Since the summation is a sum of products of geometric series, we know it can only have simple poles at the hypersurfaces defined by the denominators in (13.1.2), such as $1-q^{-\frac{1}{2}}x_{12}x_{13}x_{14}$, etc. We need to show that the residue vanishes along:

- zeroes of the factors involving 4 different $x_{ij}$s, such as $1-q^{-1}x_{13}x_{14}x_{23}x_{24} = 0$, as well as
- zeroes of factors coming from $L(1, \mathrm{ad})$, i.e. $1 \pm q^{-\frac{1}{2}}x_{ij}^{\pm} = 0$.

These are checked by routine computer computation (see remark below for some discussion of why this is easier than just checking the original result directly). ∎

*Remark* 13.1.6. Although the computation cost is highly dependent on the algorithm and implementation, it is reasonable to expect that the residues are significantly easier to handle computationally, because the "length" of the symbolic expression, in a vague sense, is at least expected to be between 1/16 to 1/24 of that of the full tetrahedral symbol: for example, there are $2^4 = 16$ cases in $1 - q^{-1} x_{\mathbf{13}}^{\pm} x_{\mathbf{14}}^{\pm} x_{\mathbf{23}}^{\pm} x_{\mathbf{24}}^{\pm}$, and $2^2 \times 6 = 24$ cases in $1 \pm q^{-\frac{1}{2}} x_{ij}^{\pm}$. On the other hand, the expression is mainly products of two-term polynomials, therefore the time complexity of the computation can potentially grow exponentially with respect to the length.

**13.2. The dual side.** On the dual side, we consider $\mathrm{Spin}_{12}$, and one of its two half-spin representations (the one relevant to us is the last fundamental representation of highest weight $\varpi_6$). We will now describe in more detail the the Lagrangian P in the half-spin representation $S^{\mathrm{ev}}$ defined by the pure spinors.

**13.2.1.** Indeed, equip $\mathbb{C}^{12}$ with the anti-diagonal bilinear form

$$Q(x, y) = \sum_{i=1}^{12} x_i y_{12-i+1}, \tag{13.2.1}$$

so that it decomposes into the direct sum of two maximal isotropic spaces

$$\mathbb{C}^{12} = V_6 \oplus V_6^*,$$

where $V_6$ is the first six coordinates. The exterior algebra $\wedge^\bullet V_6$ is a module of the Clifford algebra $\mathrm{Cl}(\mathbb{C}^{12}, Q)$ (the quotient of the tensor algebra of $\mathbb{C}^{12}$ by the relation $x \otimes y + y \otimes x = Q(x, y)$). The action of $\mathrm{Cl}(\mathbb{C}^{12}, Q)$ is as follows: vectors in $V_6$ act by wedging, and vectors in $V_6^*$ act by contracting. It is not hard to see (by fixing bases in $V_6$ and $V_6^*$) that this induces an isomorphism of associative algebras

$$\mathrm{Cl}(\mathbb{C}^{12}, Q) \simeq \mathrm{End}(\wedge^\bullet V_6).$$

Since up to isomorphism $\wedge^\bullet V_6$ is the unique simple module of $\mathrm{End}(\wedge^\bullet V_6)$, we see that as an abstract $\mathrm{Cl}(\mathbb{C}^{12}, Q)$-module this construction is independent of choice of the splitting (13.2.1) up to isomorphism. Consequently, the automorphisms of $(\mathbb{C}^{12}, Q)$ act projectively upon it, and this actually lifts to a genuine action of $\mathrm{Spin}_{12}$ — this is the 64-dimensional spin representation. The spin representation decomposes into two 32-dimensional half-spin representations $S^{\mathrm{ev}}$ and $S^{\mathrm{odd}}$, that is, the subspace of even- and odd- degree elements; in the labeling of Bourbaki [Bou02, Plate IV] these are respectively the representations of highest weight $\varpi_6$ and $\varpi_5$.

**13.2.2.** The distinguished element

$$1 \in \mathbb{C} = \wedge^0 V_6$$

which we will denote by $\mathbf{v}_0$ for better visibility, is annihilated by the subspace $V_6^*$ under the Clifford action, and one readily verifies that this characterizes it up to scalars. From the uniqueness claim above, it follows that *any* Lagrangian (i.e., maximal Q-isotropic) subspace of $\mathbb{C}^{12}$ annihilates a one-dimensional subspace in $\wedge^\bullet V_6$ under the Clifford action. A *pure spinor* is any vector in $\wedge^\bullet V_6$ belonging to such a line; thus $\mathbf{v}_0$ is a pure spinor.

Clearly, pure spinors form a cone $P_\pm$. Let $P_\pm^\times$ be the open subset of nonzero pure spinors. Then the quotient of $P_\pm^\times$ by scaling, i.e. the associated projective subvariety of the projectivization of $\wedge^\bullet V_6$, is evidently identified with the space of *all* Lagrangian subspaces of $\mathbb{C}^{12}$, that is to say, the Lagrangian Grassmannian $\mathsf{LGr}(\mathbb{C}^{12})$ of $\mathbb{C}^{12}$. This Lagrangian Grassmanian splits into two orbits under $SO_{12}$, i.e.

$$\mathsf{LGr}(\mathbb{C}^{12}) = \mathsf{LGr}^+ \coprod \mathsf{LGr}^-.$$

In fact, two isotropic subspaces $A, B$ belong to the same orbit if and only if the dimension of $A \cap B$ is even.

One gets, therefore, a corresponding decomposition

$$P_\pm = P_+ \cup P_-$$

of the cone of spinors into sub-cones that intersecgt precisely at the origin. In fact, one readily verifies from the description above that $P_+$ is precisely the subcone of pure spinors in $S^{\mathrm{ev}} \oplus 0 \simeq S^{\mathrm{ev}}$ and $P_-$ is the cone of pure spinors in $0 \oplus S^{\mathrm{odd}} \simeq S^{\mathrm{odd}}$. Indeed, using the $\mathrm{Spin}_{12}$-action, it is enough to verify this for a single point of $P_+$ and a single point of $P_-$, which one does by explicit computation.

Since we mostly care about the cone $P_+$, we denote it simply by $P$.

**Lemma 13.2.3.** *The vector space $\mathbb{C}[P]_n$ of degree $n$ homogeneous functions on $P$ is identified with the irreducible representation of $\mathrm{Spin}_{12}$ of highest weight $n\varpi_6$.*

*Proof.* Let $P^\times$ be the nonzero elements of $P$. Since the origin has codimension $\geqslant 2$, "Hartogs's theorem" implies that any regular function on $P^\times$ extends to a regular fnction on $P$. Therefore it suffices to compute the regular function on $P^\times$. However, as we have seen above, $P^\times$ is the total space of a line bundle $L$ over the flag variety $\mathsf{LGr}^+$ minus the zero section.[16] By the Borel–Weil theorem, sections of line bundles on flag varieties are highest weight representations, with highest weight determined by the isotropy representation. This shows that $\mathbb{C}[P]_n$ is the highest weight representation of weight $n\nu$ for *some* $\nu$. To compute $\nu$ it is easiest to note that $\mathbb{C}[P]_1$ is by definition a quotient of the irreducible representation $S^{\mathrm{ev}}$, and so is in fact $S^{\mathrm{ev}}$; thus $\nu$ is the highest weight of $S^{\mathrm{ev}}$. ∎

**13.2.4.** *Poles of the trace on the spinor cone.* Lemma 13.2.3 permits us to compute the character of $\mathbb{C}[P]$ by means of the Weyl character formula:

$$\mathrm{Tr}(q^{-\frac{1}{2}}\sigma, \mathbb{C}[P]) = \frac{\sigma^\rho}{\prod_{\alpha>0}(\sigma^\alpha - 1)} \sum_{w\in W} \frac{(-1)^{\ell(w)}\sigma^{w(\rho)}}{1 - q^{-\frac{1}{2}}\sigma^{w(\varpi_6)}}.$$

Note that this function is $W$-invariant, and so none of the term $\sigma^\alpha - 1$ contributes to a pole (because you can conjugate any given $\alpha$ away). Therefore, the poles are only given by $1 - q^{-\frac{1}{2}}\sigma^{w(\varpi_6)}$. We see that the hypersurfaces supporting those poles coincide with the

[16] The square has a nice description: $L^{\otimes(-2)}$ is the pullback of the determinant bundle over $\mathsf{LGr}^+$.

ones on the automorphic side. Moreover, just as in Remark 13.1.6, it is easy to see that the residues are significantly simpler expressions: there are 32 weights in $S$, so the symbolic length of a residue is only 1/32 of the full trace.

**13.2.5.** We compute the behavior of the trace when $\sigma = t^{\mu}$ for some coweight $\mu$. It suffices to assume that $\mu$ is dominant by $W$-invariance, and since it is enough to consider *generic* directions, we assume that $\mu$ is *strictly* dominant. Looking at each summand in the trace, and we want to show that for any $w \in W$, the limit of

$$\left[\frac{1}{\prod_{\alpha>0}(t^{\langle\alpha,\mu\rangle}-1)}\right]\left[\frac{t^{\langle\rho+w(\rho),\mu\rangle}}{1-q^{-\frac{1}{2}}t^{\langle w(\varpi_6),\mu\rangle}}\right]=\left[\frac{1}{\prod_{\alpha>0}(1-t^{-\langle\alpha,\mu\rangle})}\right]\left[\frac{t^{\langle-\rho+w(\rho),\mu\rangle}}{1-q^{-\frac{1}{2}}t^{\langle w(\varpi_6),\mu\rangle}}\right] \tag{13.2.2}$$

when $t \to 0$ or $\infty$ is bounded.

We first consider the case $t \to 0$. In this case, we use the left-hand side of (13.2.2). Since $\mu$ is strictly dominant, then we have $(t^{\langle\alpha,\mu\rangle}-1) \to -1$ for any positive root $\alpha$, and

$$t^{\langle\rho+w(\rho),\mu\rangle} \longrightarrow 0 \text{ or } 1,$$

because $\rho+w(\rho)$ is a sum of positive roots. Lastly, the denominator $1-q^{-\frac{1}{2}}t^{\langle w(\varpi_6),\mu\rangle}$ goes to either 1 or $1-q^{-\frac{1}{2}}$ or $\infty$, and so $\mathrm{Tr}(q^{-\frac{1}{2}}\sigma, \mathbb{C}[P])$ stays uniformly bounded when $t \to 0$. The case $t \to \infty$ is similarly proved by using the right-hand side of (13.2.2).

**13.3. Comparison using residues.** We have now located the poles of both sides of (13.0.1). They are, as we have seen, located on the locus where an eigenvalue of $\sigma$ on $S$ coincides with $q^{\frac{1}{2}}$. The corresponding residues, on either side, can be computed using a computer as well — this computation is substantially smaller than computing the full expressions — and it then turns out these residues can be explicitly factorized.

For example, we record the residue of both sides of (13.0.1) when $1-q^{-\frac{1}{2}}\sigma^{-\varpi_6} = 1-q^{-\frac{1}{2}}(x_{12}x_{13}x_{14})^{-1}=0$:

$$\begin{aligned}-L(1,\mathrm{ad})x_{23}^4x_{24}^4x_{34}^4\Big[&(x_{12}^2-1)(x_{13}^2-1)(x_{14}^2-1)(x_{12}x_{13}x_{24}-x_{34})\\ &(x_{12}x_{13}x_{34}-x_{24})(x_{12}x_{13}-x_{24}x_{34})(x_{12}x_{13}x_{24}x_{34}-1)(x_{12}x_{14}x_{23}-x_{34})\\ &(x_{12}x_{14}x_{34}-x_{23})(x_{12}x_{14}-x_{23}x_{34})(x_{12}x_{14}x_{23}x_{34}-1)(x_{13}x_{14}x_{23}-x_{24})\\ &(x_{13}x_{14}x_{24}-x_{23})(x_{13}x_{14}-x_{23}x_{24})(x_{13}x_{14}x_{23}x_{24}-1)\Big]^{-1},\end{aligned}$$

in which we really meant to replace, for example, $x_{12}$ by $q^{-\frac{1}{2}}(x_{13}x_{14})^{-1}$ (so that $q^{-\frac{1}{2}}$ is not treated as a variable but a constant); however, replacing $q^{-\frac{1}{2}}$ by $x_{12}x_{13}x_{14}$ makes the expression look more symmetric.

This completes the proof, according to the general plan outlined at the start of the section.

## 14. Hypergeometric evaluations of the tetrahedral symbol

Our goal here is to express the tetrahedral symbol in terms of generalized hypergeometric functions, proving both Proposition 7.3.1 and the formulas given in § 7.4.

**14.1. Classical hypergeometric functions and Mellin–Barnes integrals.** For positive integers $k < l$ and parameters $\underline{a} = (a_1, \dots, a_l) \in \mathbb{C}^l$ and $\underline{b} = (b_1, \dots, b_k) \in \mathbb{C}^k$, the generalized hypergeometric function of parameters $\underline{a}$ and $\underline{b}$ is the analytic continuation of the series

$$ {}_lF_k(\underline{a}, \underline{b}; x) = {}_lF_k\left( \begin{matrix} a_1, \dots, a_l \\ b_1, \dots, b_k \end{matrix} \,\middle|\, x \right) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{l} (a_j)_n}{\prod_{j=1}^{k} (b_j)_n} \frac{x^n}{n!}, $$

where $(a)_n$ is the *rising* factorial:

$$ (a)_0 := 1, \quad (a)_n := a(a+1)\cdots(a+n-1). $$

Classical 6j symbols (i.e., compact $\mathcal{R}$ and $F = \mathbb{R}$) enjoy various interpretations at the value at the singular point $x = 1$ of ${}_4F_3(\underline{a}, \underline{b}; x)$ for certain $\underline{a}$ and $\underline{b}$; our goal is to describe a similar result for the tetrahedral symbols for $F = \mathbb{R}$.

Using Mellin transform and its inverse, one can rewrite ${}_lF_k$ into an *Mellin–Barnes* type integral (with some assumptions on the parameters):

$$ \frac{\Gamma(a_1)\cdots\Gamma(a_l)}{\Gamma(b_1)\cdots\Gamma(b_k)} {}_lF_k(\underline{a}, \underline{b}; x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(a_1+s)\cdots\Gamma(a_l+s)}{\Gamma(b_1+s)\cdots\Gamma(b_k+s)} \Gamma(-s)(-x)^s ds, $$

where the vertical line $(c - i\infty, c + i\infty)$ separates the poles of all $\Gamma(a_j + s)$ from those of $\Gamma(-s)$.[17] One can readily generalize the notion of Mellin–Barnes integrals to any local field because $\Gamma$-functions are essentially the local L-factors over $\mathbb{R}$ (cf. § 2). Our procedure will, in fact, be to first derive a Mellin–Barnes representation (in this generalized sense) of the tetrahedral symbol in the principal series case, and then derive the various desired consequences from it.

**14.2. Review on Mellin transforms.** We review the properties of Mellin transforms over an arbitrary local field.

**Lemma 14.2.1.** *Suppose* $a, b > 0$ *and* $a + b < 1$, *then we have equality*

$$ \int_F |x|^{a-1} |1-x|^{b-1} dx = \frac{\gamma(a+b)}{\gamma(a)\gamma(b)}, $$

*where the left-hand side is absolutely convergent.*

*Proof.* Since $a + b - 2 < -1$ (resp. $a - 1 > -1$, resp. $b - 1 > -1$), the integral is absolutely convergent near $\infty$ (resp. $0$, $1$). Thus the whole integral is absolutely convergent. For the equality, we note that

$$ \begin{aligned} \int_F |x|^{a-1}|1-x|^{b-1} dx \cdot \gamma(a+b)^{-1} &\overset{(2.2.7)}{=} \int_F |x|^{a-1}|1-x|^{b-1} dx \int_F |y|^{a+b-1} \Psi(y) dy \\ &= \int_F \left|\frac{x}{y}\right|^{a-1} \left|1 - \frac{x}{y}\right|^{b-1} d\left(\frac{x}{y}\right) \int_F |y|^{a+b-1} \Psi(y) dy \\ &= \int_F |x|^{a-1} |y-x|^{b-1} dx \int_F \Psi(y) dy, \end{aligned} $$

[17] We will not use this general fact here; in the case relevant to us, namely ${}_4F_3$, details are contained in § 14.4.

which is the Fourier transform of the convolution of $|x|^{a-1}$ and $|x|^{b-1}$. Here we used the fact that $0 < a + b < 1$ implies that the integral form of $\gamma(a+b)^{-1}$ is also absolutely convergent, hence we are free to use Fubini theorem to manipulate the integrals above. Applying (2.2.7) again we arrive at

$$\int_F |x|^{a-1}|1-x|^{b-1}dx \cdot \gamma(a+b)^{-1} = \gamma(a)^{-1}\gamma(b)^{-1},$$

and this finishes the proof. ∎

**Definition 14.2.2.** For a function $f$ on $F^\times$, we define its *Mellin transform* to be the integral

$$\mathcal{M}_f(\mu) = \int_{F^\times} f(x)\mu(x)d^\times x,$$

assuming the integral is absolutely convergent for all (unitary) characters $\mu$ of $F^\times$. We will allow ourselves to speak of the integral for quasi-characters by means of analytic continuation, when applicable. When $F = \mathbb{R}$, we denote $\mathcal{M}_f^+(s) := \mathcal{M}_f(|\cdot|^s)$ and $\mathcal{M}_f^-(s) := \mathcal{M}_f(\mathrm{sgn}|\cdot|^s)$. In the reverse direction, given a function $\widehat{f}(\chi)$ on the character group $\widehat{F^\times}$, we define its inverse Mellin transform as

$$\mathcal{M}_{\widehat{f}}^{-1}(x) = \int_{\widehat{F^\times}} \widehat{f}(\chi)\chi(x)d\chi,$$

where the measure $d\chi$ is dual to $d^\times x$, which amounts to asking that this is indeed inverse to the Mellin transform.

**Lemma 14.2.3.** *Suppose that $f$ is an $L^1$-function on $F^\times$ with the property that $\mathcal{M}_f(\mu)$ defines an $L^1$-function on the character group of $F^\times$. Then the inverse Mellin transform of $\mathcal{M}_f$ defines a continuous function that agrees with $f$ almost everywhere.*

*Proof.* This is a standard fact of harmonic analysis. In the $F = \mathbb{R}$ case, using the fact that $\mathbb{R}^\times = \mathbb{R}_{>0} \times \{\pm 1\}$ and change of variables, the lemma is a standard precise form of Fourier inversion; see for example [SW71, Corollary 1.21]. The $F = \mathbb{C}$ case can be similarly derived from the fact that $\mathbb{C}^\times = \mathbb{R}_{>0} \times S^1$. The nonarchimedean case is relatively easy to deduce because continuous functions are locally constant. It can be proved by first verifying that the inverse Mellin transform of $\mathcal{M}_f$ defines a continuous function, call it $f_1$, and then verifying that the pairings of $f$ or $f_1$ with the characteristic function of any compact open subset of $F^\times$ coincide. We leave the details to the reader. ∎

For a nonempty open interval $J \subset \mathbb{R}$, and let $L^1_J$ be the functions on $F^\times$ with the property that $\int_{F^\times} |f(x)| \cdot |x|^\sigma d^\times x < \infty$ whenever $\sigma \in J$. In practice, we will be interested in the interval $J = (0, \frac{1}{2})$. Many statements can be reduced to the case when $J$ contains zero, simply by multiplying $f$ by a suitable power of $|x|$.

**Lemma 14.2.4.** *For $f \in L^1_J$, the Mellin transform $\mathcal{M}_f(\mu_s)$ is absolutely convergent for a character $\mu$ and any $s$ whose real part belongs to $J$.*

When we write $\mathcal{M}_f$ for $f \in L^1_J$, we will always regard it as a function on the set of quasi-characters specified by this Lemma. Next, $L^1_J$ behaves well with respect to convolution:

**Lemma 14.2.5.** *If* $f_1, f_2 \in L^1_J$, *the* multiplicative *convolution*

$$f(y) := \int_{F^\times} f_1(x) f_2(yx^{-1}) d^\times x$$

*is absolutely convergent for almost all* $y$, *and also defines an element of* $L^1_J$. *Moreover, the Mellin transforms are multiplicative: we have*

$$\mathcal{M}_f = \mathcal{M}_{f_1}\mathcal{M}_{f_2},$$

*where, as noted above, we regard both sides as functions on quasi-characters* $\mu_s$ *where the real part of* $s$ *lies in* $J$.

We omit the straightforward proof.

**Lemma 14.2.6.** *For two characters* $\alpha$ *and* $\beta$ *of* $F^\times$, *the product* $\alpha(x)\beta_-(1-x)$ *belongs to the space* $L^1_{(0,\frac{1}{2})}$ *defined in Lemma 14.2.5. Its Mellin transform is given by*

$$\mathcal{M}_{\alpha(x)\beta_-(1-x)}(\mu_s) = \frac{\gamma(\alpha\beta_+\mu_s)}{\gamma(\alpha\mu_s)\gamma(\beta_+)},$$

*valid for* $\Re(s) \in (0, \frac{1}{2})$.

*Proof.* The proof of Lemma 14.2.1 still works: the real part of the exponent of quasi-character $\alpha\mu_s$ is between $0$ and $\frac{1}{2}$, while that of $\beta_+$ is $\frac{1}{2}$, and so the sum of two exponents has real part between $0$ and $1$; this way the relevant integrals all converge absolutely. ∎

**14.3. Hypergeometric functions over local fields.** The edge integral we have described in (7.2.1), up to scaling by powers of $\nu_{\mathbb{P}}$, has the following alternative form by letting $w = 1$ (see § 11 for how the measure is properly dealt with):

$$I = \int_{F^3} \alpha_{1-}(x)\alpha_{2-}(1-x)\alpha_{3-}(x-y)\alpha_{4-}(y-z)\alpha_{5-}(y)\alpha_{6-}(z)\alpha_{7-}(z-1),$$

for certain characters $\alpha_i : F^\times \to \mathbb{C}^\times$, and the measure is the usual additive Haar measure $dxdydz$, which we omit for simplicity. We will explain how to evaluate this integral in terms of a generalized hypergeometric function. Note that we have already proven that $I$ is absolutely convergent, and thereby, by Fubini's theorem, it can be evaluated as an iterated integral, in any way we please.

Use the shorthand $\alpha_{12} = \alpha_1\alpha_2$, etc., and rewrite the integral as

$$I = \int_{F^3} \alpha_{13--}(x)\alpha_{2-}(1-x)\alpha_{3-}(1-y/x)\alpha_{4-}(1-z/y)\alpha_{45--}(y)\alpha_{67--}(z)\alpha_{7-}(1-1/z).$$

We are going to repeatedly apply Lemma 14.2.5 with the interval $J$ taken to be $(0, \frac{1}{2})$. First of all, we take $f_1(x) = \alpha_{13}(x)\alpha_{2-}(1-x)$ and $f_2(x) = \alpha_{3-}(1-x)$; they both belong to $L^1_J$ by Lemma 14.2.6, and therefore their multiplicative convolution

$$f_{123}(y) := \int_F \alpha_{13--}(x)\alpha_{2-}(1-x)\alpha_{3-}(1-y/x)dx \left(= \int_F \alpha_{1-}(x)\alpha_{2-}(1-x)\alpha_{3-}(x-y)dx\right).$$

defines also a class in $L^1_J$. By Lemma 14.2.5 and Lemma 14.2.6 we have

$$\mathcal{M}_{f_{123}}(\mu_s) = \frac{\gamma(\alpha_{13}\alpha_{2+}\mu_s)}{\gamma(\alpha_{13}\mu_s)\gamma(\alpha_{2+})} \cdot \frac{\gamma(\alpha_{3+}\mu_s)}{\gamma(\mu_s)\gamma(\alpha_{3+})}.$$

where we assume that the real part of $s$ belongs to $(0, \frac{1}{2})$; we will continue to impose this assumption below.

Multiplication by a unitary character of course preserves the property of belonging to $L^1_J$. Consequently, we can apply Lemma 14.2.5 to analyze the multiplicative convolution $f_{12345}(z)$ of $f_{123}(y) \cdot \alpha_{45}(y)$ and $\alpha_{4-}(1-y)$. Then $f_{12345}$ belongs to $L^1_J$ and its Mellin transform is given by

$$\begin{aligned}\mathcal{M}_{f_{12345}}(\mu_s) &= \mathcal{M}_{f_{123}(z)\alpha_{45}(z)}(\mu_s)\mathcal{M}_{\alpha_{4-}(1-z)}(\mu_s)\\ &= \frac{\gamma(\alpha_{12345+}\mu_s)}{\gamma(\alpha_{1345}\mu_s)\gamma(\alpha_{2+})} \cdot \frac{\gamma(\alpha_{345+}\mu_s)}{\gamma(\alpha_{45}\mu_s)\gamma(\alpha_{3+})} \cdot \frac{\gamma(\alpha_{4+}\mu_s)}{\gamma(\mu_s)\gamma(\alpha_{4+})}.\end{aligned}$$

Finally, multiplying $f_{12345}(z)$ by $\alpha_{67}(z)$ and then convolving with $\alpha_{7-}(1-z)$ gives, for exactly the same reason as before, a function $f_{1234567}(w) \in L^1_J$ such that

$$\begin{aligned}\mathcal{M}_{f_{1234567}}(\mu_s) &= \mathcal{M}_{f_{123}(w)\alpha_{4567}(w)}(\mu_s)\mathcal{M}_{\alpha_{67}(w)\alpha_{4-}(1-w)}(\mu_s)\mathcal{M}_{\alpha_{7-}(1-w)}\\ &= \frac{\gamma(\alpha_{1234567+}\mu_s)}{\gamma(\alpha_{134567}\mu_s)\gamma(\alpha_{2+})} \cdot \frac{\gamma(\alpha_{34567+}\mu_s)}{\gamma(\alpha_{4567}\mu_s)\gamma(\alpha_{3+})} \cdot \frac{\gamma(\alpha_{467+}\mu_s)}{\gamma(\alpha_{67}\mu_s)\gamma(\alpha_{4+})} \cdot \frac{\gamma(\alpha_{7+}\mu_s)}{\gamma(\mu_s)\gamma(\alpha_{7+})},\end{aligned}$$

which we rewrite as

$$\left[\gamma(\alpha_{2+})\gamma(\alpha_{3+})\gamma(\alpha_{4+})\gamma(\alpha_{7+})\right]^{-1} \frac{\gamma(\alpha_{1234567+}\mu_s)\gamma(\alpha_{34567+}\mu_s)\gamma(\alpha_{467+}\mu_s)\gamma(\alpha_{7+}\mu_s)}{\gamma(\alpha_{134567}\mu_s)\gamma(\alpha_{4567}\mu_s)\gamma(\alpha_{67}\mu_s)\gamma(\mu_s)}. \tag{14.3.1}$$

Note that $f_{1234567}(w)$ is defined by replacing $\alpha_{7-}(z-1)$ by $\alpha_{7-}(z-w)$ in the definition of I. Therefore, the edge integral I is the value at 1 of a function $f_{1234567}$ whose Mellin transform is given by (14.3.1).

Moreover, the Mellin inversion formula is applicable, in the following form:

$$I = f_{1234567}(1) = \int \mathcal{M}_{f_{1234567}}(\mu_s)d\mu, \tag{14.3.2}$$

where $0 < s < \frac{1}{2}$ is any fixed number, and the integral is taken over all characters $\mu$ (see § 2.1 for the measure on $\mu$). To verify the applicability of the Mellin inversion formula, we verify that $\mathcal{M}_{f_{1234567}}(\mu_s)$ is absolutely integrable as a function of $\mu$ and invoke Lemma 14.2.3. We will check this absolute integrability in the nonarchimedean case, leaving the archimedean cases to the reader; a similar check for $F = \mathbb{R}$ is carried out after (14.4.2).

Suppose, then, that the cardinality of the residue field of F equals q. Let $\mathcal{O} \subset F$ be the ring of integers. The decomposition $F^\times \simeq \mathcal{O}^\times \times \mathbb{Z}$ (after fixing a uniformizer) gives a corresponding decomposition

$$\widehat{F^\times} \simeq \widehat{\mathcal{O}^\times} \times S^1.$$

As usual, we say that a character $\mu$ of $\mathcal{O}^\times$ has conductor $n$ if it is trivial on $1 + \varpi^n\mathcal{O}$ but not on any larger subgroup of this form. The number of such characters equals $q(1-2/q)$ for

$n = 1$ and $q^n(1 - 1/q)^2$ for $n > 1$; all that matters is that it is $O(q^n)$. On the other hand, the formulas of § 2.2.3 imply that the the absolute value of the term

$$\frac{\gamma(\alpha_{1234567+}\mu_s)\gamma(\alpha_{34567+}\mu_s)\gamma(\alpha_{467+}\mu_s)\gamma(\alpha_{7+}\mu_s)}{\gamma(\alpha_{134567}\mu_s)\gamma(\alpha_{4567}\mu_s)\gamma(\alpha_{67}\mu_s)\gamma(\mu_s)}$$

from (14.3.1) equals $q^{-2n}$ when $\mu$ has conductor $n$, where $n$ is chosen strictly larger than the conductor of any $\alpha_i$ (this ensures that the conductor of any $\alpha_\bullet\mu$ involved in the fraction above equals the conductor of $\mu$). Therefore, the integral of (7.2.2) is absolutely bounded by a constant multiple of $\sum_{n \geqslant 0} q^n \cdot q^{-2n}$ and is absolutely convergent. This concludes our justification of (14.3.2).

**14.4. Relationship with classical $_4F_3$.** When $F = \mathbb{R}$, we may use (14.3.1) to relate the tetrahedral symbol for the principal series with generalized hypergeometric functions.

For simplicity, we assume characters $\alpha_{1234567+}$, etc. in the numerator of (14.3.1) and $\alpha_{134567}$, etc. in the denominator are of the forms $|\cdot|^{a_j}$ and $|\cdot|^{b_j}$ ($j = 1, \dots, 4$) respectively, where

$$a_j \in \frac{1}{2} + i\mathbb{R}, \quad b_j \in i\mathbb{R},$$

which is the case that is relevant to evaluating the tetrahedral symbol for unramified characters with $F = \mathbb{R}$. The general case can be analyzed similarly, where one adds various signs; the answer itself will look different because the contours of integration used later in the argument need to be chosen differently.

The Mellin transform over $\mathbb{R}^\times$ consists of two disjoint components $\mathcal{M}^+$ and $\mathcal{M}^-$ (see Definition 14.2.2). For a function $f$ on $\mathbb{R}^\times$, write $f = f^+ + f^-$, where $f^+$ is even and $f^-$ is odd, then we have

$$\mathcal{M}^-_{f^+} = \mathcal{M}^+_{f^-} = 0.$$

Moreover, $\mathcal{M}^+_{f^+}$ (resp. $\mathcal{M}^-_{f^-}$) is twice the classical Mellin transform of $f^+|_{(0,\infty)}$ (resp. $f^-|_{(0,\infty)}$). As a result, the sum $\mathcal{M}^+_f + \mathcal{M}^-_f$ is twice the classical Mellin transform of the function $f|_{(0,\infty)}$.

Therefore, to recover the value of $f = f_{1234567}$ at 1, we can use the inverse Mellin transform on the sum of

$$\mathcal{M}^+_f(s) = \prod_{j=1}^{4} \frac{\gamma(a_j + s)}{\gamma(a_j - b_j)\gamma(b_j + s)}$$

and

$$\mathcal{M}^-_f(s) = \prod_{j=1}^{4} \frac{\gamma^-(a_j + s)}{\gamma(a_j - b_j)\gamma^-(b_j + s)},$$

where $\gamma^-(\mu)$ means $\gamma(\mu \cdot \mathrm{sgn})$ for any quasi-character $\mu$.

We will now use the relations from (2.2.2):

$$\gamma^\pm(s)^{-1} = \Gamma(s)(\mathbf{I}^{-s} \pm \bar{\mathbf{I}}^{-s}) = (2\pi)^{-s}\Gamma(s)(i^{-s} \pm i^s),$$
$$\gamma^\pm(s) = \Gamma(1 - s)(\mathbf{I}^{s-1} \pm \bar{\mathbf{I}}^{s-1}) = (2\pi)^{s-1}\Gamma(1 - s)(i^{s-1} \pm i^{1-s})$$

with $\mathbf{I} := 2\pi i$ and $i^s := e^{i\pi s/2}$, to rewrite $\mathcal{M}_f^+$ and $\mathcal{M}_f^-$ in terms of $\Gamma$-functions and exponential functions:

$$\mathcal{M}_f^{\pm} = \underbrace{\prod_{j=1}^{4} \frac{(2\pi)^{a_j-b_j-1}}{\gamma(a_j-b_j)}}_{=:A} \times \prod_{j=1}^{4} \Gamma(b_j+s)\Gamma(1-a_j-s) \underbrace{\prod_{j=1}^{4} (i^{-b_j-s} \pm i^{b_j+s})(i^{a_j+s-1} \pm i^{1-a_j-s})}_{=:B_{\pm}}$$

Therefore, to compute $\mathcal{M}_f^+(s) + \mathcal{M}_f^-(s)$, we expand

$$A(B_+ + B_-) = \sum_{k=-4}^{4} C_k i^{2ks} = \sum_{k=-2}^{2} C_{2k} i^{4ks} \tag{14.4.1}$$

in powers of $i^s$, where the $C_k$s are various constants that are sums of products of $(2\pi)^{a_j-b_j-1}$, $i^{\pm a_j}$, $i^{\pm b_j}$ and $\gamma(a_j-b_j)$; for the last equality, it is easy to see that $C_k = 0$ for all odd $k$ because terms from $B_+$ and from $B_-$ cancel.

Now fix $0 < \sigma < \frac{1}{2}$. Taking classical inverse Mellin transform, we have

$$f(x) = \sum_{k=-2}^{2} \frac{C_{2k}}{4\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \prod_{j=1}^{4} \Gamma(b_j+s)\Gamma(1-a_j-s) i^{4ks} x^{-s} ds \tag{14.4.2}$$

where we take the straight line contour from $-\sigma - i\infty$ to $-\sigma + i\infty$. For simplicity, we will restrict to $x \in \mathbb{R}_{>0}$, since we will be most interested in the value at $x = 1$; in particular, power functions are well-defined.

In order to justify the application of inverse Mellin transform, observe that the integral is indeed absolutely convergent. To handle the asymptotics, it is convenient to rewrite, using the relation $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$, the product of $\Gamma$-functions as

$$\prod_{j=1}^{4} \frac{\Gamma(1-a_j-s)}{\Gamma(1-b_j-s)} \frac{\pi}{\sin(\pi(b_j+s))} \tag{14.4.3}$$

and use the fact ([TE51]) that the ratio $\Gamma(s+a)/\Gamma(s+b)$ is asymptotic to $s^{a-b}$ so long as we restrict the argument of $s$ to be within $(-\pi+\delta, \pi-\delta)$ for any fixed $\delta > 0$.[18] In particular, when $s$ is restricted to any vertical line, this same ratio is bounded by $(1+|t|)^{\Re(a)-\Re(b)}$ where $t$ is the imaginary part of $s$. Consequently, the integrand has the asymptotic behavior $|t|^{-2}$, and so is absolutely convergent. Changing variables $s \mapsto -s, k \mapsto -k$, we obtain

$$f(x) = \sum_{k=-2}^{2} \frac{C_{-2k}}{4\pi i} \int_{-\sigma-i\infty}^{-\sigma+i\infty} \prod_{j=1}^{4} \Gamma(b_j-s)\Gamma(1-a_j+s) i^{4ks} x^{s} ds. \tag{14.4.4}$$

The integral

$$G_{4,4}^{4,4}\left( \begin{matrix} a_1, \dots, a_4 \\ b_1, \dots, b_4 \end{matrix} \,\middle|\, x \right) := \frac{1}{2\pi i} \int_L \prod_{j=1}^{4} \Gamma(b_j-s)\Gamma(1-a_j+s) x^s ds \tag{14.4.5}$$

[18] That is to say, $s^{b-a}\Gamma(s+a)/\Gamma(s+b)$ approaches 1 as $s$ approaches infinity in this region.

taken along certain admissible paths L, is known as a Meijer G-function. To understand its connection with generalized hypergeometric functions, we need to review some of its properties. The general theory of G-functions is rich, but what we need can be easily derived from basic complex analysis.

For our choice of L, namely, the vertical contour with real part $-\sigma$, all the poles of $\Gamma(1-a_j+s)$ appears to the left of L and those of $\Gamma(b_j-s)$ to the right. We then use residues to evaluate the integral; but we will shift the contour in different ways according to whether $|x|$ is less than unity or greater than unity.

We first assume that $|x| \leqslant 1$. In this case, we shift the integral to the line where the real part of $x$ equals some large positive real $c$, where $c$ is chosen ("pole avoidance") so that its distance from the series of points $b_j, b_j+1, \dots$ is at least $1/8$ (which is possible because $j$ ranges from 1 to 4). In order to shift contours in this way, we consider the segment of our integral from $-\sigma-iT$ to $-\sigma+iT$ and connect it to the line segment from $c-iT$ to $c+iT$ by means of horizontal segments. The same reasoning that is given after (14.4.3) shows that, as we take $T\to\infty$, the contribution of both horizontal segments vanish. Moreover, if we then take $c\to\infty$, the contribution of the right-hand segment vanishes too; here we use both the fact that $|x|\leqslant 1$ and the uniformity of the asymptotic for $\frac{\Gamma(s+a)}{\Gamma(s+b)}$ in the relevant region. The importance of the choice of $c$ is to ensure that the term $\sin(\pi(b_j-s))$ is bounded away from zero.

This shows that when $|x|\leqslant 1$, the integral (14.4.5) is equal to the *negative* of the series whose terms are the residues at all $b_j+n$, in the sense that one converges absolutely if and only if the other does. More explicitly, this series is

$$\begin{aligned}
&-\sum_{n=0}^{\infty}\sum_{h=1}^{4}\frac{(-1)^n\prod_{j=1}^4\Gamma(1-a_j+b_h+n)\prod_{j\neq h}\Gamma(b_j-b_h-n)}{n!}x^{b_h+n}\\
&\quad=-\sum_{h=1}^{4}\prod_{j=1}^{4}\Gamma(1-a_j+b_h)\prod_{j\neq h}\Gamma(b_j-b_h)x^{b_h}\times\sum_{n=0}^{\infty}\frac{\prod_{j=1}^4(1-a_j+b_h)_n}{\prod_{j\neq h}(1-b_j+b_h)_n}\frac{x^n}{n!}.
\end{aligned}$$

Using the definition of ${}_4F_3$, we then obtain

$$\begin{aligned}
G^{4,4}_{4,4}\left(\begin{matrix}a_1,\dots,a_4\\ b_1,\dots,b_4\end{matrix}\,\middle|\,x\right)&=-\sum_{h=1}^{4}\prod_{j=1}^{4}\Gamma(1-a_j+b_h)\prod_{j\neq h}\Gamma(b_j-b_h)x^{b_h}\\
&\qquad\times{}_4F_3\left(\begin{matrix}1+b_h-a_j, j=1,\dots,4\\ 1+b_h-b_j, j\neq h\end{matrix}\,\middle|\,x\right).
\end{aligned}$$

The series defining these ${}_4F_3$ converge absolutely when $|x|\leqslant 1$, and so the whole equality is valid in the same domain.

Similarly, when $|x|\geqslant 1$, we shift the contour to the *left*, i.e. choose $c$ to be very negative, now incurring poles when $s=a_h-1-n$ for $n\geqslant 0$. The resulting formula is

$$G^{4,4}_{4,4}\left(\begin{matrix}a_1,\dots,a_4\\ b_1,\dots,b_4\end{matrix}\,\middle|\,x\right)=\sum_{h=1}^{4}\prod_{j=1}^{4}\Gamma(1-a_h+b_j)\prod_{j\neq h}\Gamma(a_h-a_j)x^{a_h-1}$$

$$\times\ {}_4F_3\left(\begin{matrix} 1-a_h+b_j, j=1,\dots,4 \\ 1-a_h+a_j, j\neq h\end{matrix}\,\middle|\, x^{-1}\right).$$

This in particular also allows us to analytically continue the G-function for all $x$ within our chosen domain containing 1.

Finally, we go back to (14.4.4). For the summand where $k=0$, the integral involved is exactly the Meijer G-function discussed above. For any $k\neq 0$, we may use the same contour integral argument and the fact that $i^{4kn}=1$ for any $n\in\mathbb{Z}$, and see that (we use the $|x|\leqslant 1$ formula here for example)

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty}\prod_{j=1}^{4}\Gamma(b_j-s)\Gamma(1-a_j+s)i^{4ks}x^s ds\\ &\quad=-\sum_{h=1}^{4}\prod_{j=1}^{4}\Gamma(1-a_j+b_h)\prod_{j\neq h}\Gamma(b_j-b_h)i^{4kb_h}x^{b_h}\times\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{4}(1-a_j+b_h)_n}{\prod_{j\neq h}(1-b_j+b_h)_n}\frac{x^n}{n!}.\end{aligned}$$

Combining everything together, we have for $|x|\leqslant 1$:

$$f(x)=\frac{1}{2}\sum_{h=1}^{4}B_h(x)\,{}_4F_3\left(\begin{matrix}1+b_h-a_j, j=1,\dots,4\\ 1+b_h-b_j, j\neq h\end{matrix}\,\middle|\, x\right),$$

where

$$B_h(x)=-x^{b_h}\prod_{j=1}^{4}\Gamma(1-a_j+b_h)\prod_{j\neq h}\Gamma(b_j-b_h)\left(\sum_{k=-2}^{2}C_{-2k}i^{4kb_h}\right).$$

Recall by (14.4.1), we have

$$\sum_{k=-2}^{2}C_{-2k}i^{4kb_h}=A(B_++B_-)|_{s=-b_h}.$$

But for $B_-$, we have

$$B_-|_{s=-b_h}=\prod_{j=1}^{4}(i^{-b_j+b_h}-i^{b_j-b_h})(i^{a_j-b_h-1}-i^{1-a_j+b_h})=0$$

because when $j=h$ the factor

$$i^{-b_h+b_h}-i^{b_h-b_h}=0.$$

Therefore, only the term $AB_+$ survives, and so we can simplify:

$$\begin{aligned}B_h(x)&=-2x^{b_h}\prod_{j=1}^{4}\frac{(2\pi)^{a_j-b_j-1}}{\gamma(a_j-b_j)}\prod_{j=1}^{4}\Gamma(1-a_j+b_h)(i^{a_j-b_h-1}+i^{1-a_j+b_h})\\ &\qquad\prod_{j\neq h}\Gamma(b_j-b_h)(i^{-b_j+b_h}+i^{b_j-b_h})\\ &\overset{(2.2.2)}{=}-2x^{b_h}\prod_{j=1}^{4}{}'\frac{\gamma(a_j-b_h)}{\gamma(a_j-b_j)\gamma(b_j-b_h)},\end{aligned}$$

where $\prod'$ denotes that we omit any evaluations of $\gamma$ at polar points; in the case above, this means $\gamma(b_j - b_h)$ for $j = h$. As a result, we finally have for $|x| \leqslant 1$:

$$f(x) = -\sum_{h=1}^{4} x^{b_h} \left( {\prod_{j=1}^{4}}' \frac{\gamma(a_j - b_h)}{\gamma(b_j - b_h)\gamma(a_j - b_j)} \right) {}_4F_3\left( \begin{matrix} 1 + b_h - a_j, j = 1, \dots, 4 \\ 1 + b_h - b_j, j \neq h \end{matrix} \,\middle|\, x \right). \tag{14.4.6}$$

Similarly, for $|x| \geqslant 1$, we have

$$f(x) = \frac{1}{2} \sum_{h=1}^{4} A_h(x)\, {}_4F_3\left( \begin{matrix} 1 - a_h + b_j, j = 1, \dots, 4 \\ 1 - a_h + a_j, j \neq h \end{matrix} \,\middle|\, x^{-1} \right),$$

where

$$A_h(x) = x^{a_h - 1} \prod_{j=1}^{4} \Gamma(1 - a_h + b_j) \prod_{j \neq h} \Gamma(a_h - a_j) \left( \sum_{k=-2}^{2} C_{-2k} i^{4k(a_h - 1)} \right),$$

and we may simplify it as

$$f(x) = \sum_{h=1}^{4} x^{a_h - 1} \left( {\prod_{j=1}^{4}}' \frac{\gamma(a_h - b_j)}{\gamma(a_h - a_j)\gamma(a_j - b_j)} \right) {}_4F_3\left( \begin{matrix} 1 - a_h + b_j, j = 1, \dots, 4 \\ 1 - a_h + a_j, j \neq h \end{matrix} \,\middle|\, x^{-1} \right). \tag{14.4.7}$$

**14.5. The tetrahedral symbol as a convolution of $\gamma$-factors, for general $F$.** Return for a moment to the case of general $F$. We shall prove the formula Proposition 7.3.1, which we recall here:

$$\{\Pi\} = \nu_{\mathbb{P}}^{-1} \int_{\mu} \frac{\gamma(\frac{1}{2} + \epsilon, A \otimes \mu)\gamma(1 - \epsilon, B^{-1} \otimes \mu^{-1})}{\sqrt{\gamma(\frac{1}{2}, A \otimes B^{-1})}} d\mu$$

for four-element sets of characters $A, B$.

We shall apply the results of the former subsection with the following $\alpha_i$

$$\alpha_1 = \chi_{12}\chi_{14}\chi_{13}^{-1}, \quad \alpha_2 = \chi_{12}\chi_{13}\chi_{14}^{-1}, \quad \alpha_3 = \chi_{21}\chi_{24}\chi_{23}^{-1}, \quad \alpha_4 = \chi_{42}\chi_{43}\chi_{41}^{-1},$$
$$\alpha_5 = \chi_{42}\chi_{41}\chi_{43}^{-1}, \quad \alpha_6 = \chi_{41}\chi_{43}\chi_{42}^{-1}, \quad \alpha_7 = \chi_{34}\chi_{31}\chi_{32}^{-1}.$$

We compute the following combinations:

$$\alpha_{1234567} = \chi_{12}\chi_{31}\chi_{41}, \quad \alpha_{34567} = \chi_{21}\chi_{41}\chi_{31}, \quad \alpha_{467} = \chi_{43}\chi_{31}\chi_{23}, \quad \alpha_7 = \chi_{34}\chi_{31}\chi_{23},$$
$$\alpha_{134567} = \chi_{31}^2, \quad \alpha_{4567} = \chi_{42}\chi_{41}\chi_{31}\chi_{23}, \quad \alpha_{67} = \chi_{41}\chi_{31}\chi_{24}\chi_{23}.$$

We now rewrite (14.3.1) in the form

$$\left[\gamma(\alpha_{2+})\gamma(\alpha_{3+})\gamma(\alpha_{4+})\gamma(\alpha_{7+})\right]^{-1} \frac{\gamma(\frac{1}{2} + s, A' \otimes \mu)}{\gamma(s, B' \otimes \mu)}, \tag{14.5.1}$$

where we are to integrate over characters $\mu$ and a fixed real $s$ between $0$ and $\frac{1}{2}$; and

$$A' = \{\alpha_{1234567}, \alpha_{34567}, \alpha_{467}, \alpha_7\} = \chi_{21}\chi_{41}\chi_{31} \otimes \{\chi_{12}^2, 1, \chi_{43}\chi_{23}\chi_{12}\chi_{14}, \chi_{34}\chi_{23}\chi_{12}\chi_{14}\},$$
$$B' = \{\alpha_{134567}, \alpha_{4567}, \alpha_{67}, 1\} = \chi_{31}^2 \otimes \{1, \chi_{42}\chi_{41}\chi_{13}\chi_{23}, \chi_{41}\chi_{13}\chi_{24}\chi_{23}, \chi_{13}^2\}.$$

Now, (14.5.1), when integrated over $\mu$, is invariant under a common translation of $A', B'$. Doing such translating $A'$ and $B'$ both by $\chi_{13}$, we arrive at:

$$\begin{aligned} A &:= \chi_{13} \otimes A' = \big(\chi_{41} \otimes \{\chi_{12}, \chi_{21}\}\big) \cup \big(\chi_{23} \otimes \{\chi_{43}, \chi_{34}\}\big), \\ B &:= \chi_{13} \otimes B' = \{\chi_{31}, \chi_{13}\} \cup \big(\chi_{41}\chi_{23} \otimes \{\chi_{42}, \chi_{24}\}\big). \end{aligned}$$

Using the fact that $\gamma(\mu_+)\gamma(\mu_-^{-1}) = \mu(-1)$, we can replace the inverse of $\gamma(s, B \otimes \mu)$ by $\gamma(1-s, B^{-1} \otimes \mu^{-1})$. Consequently we rewrite the edge integral in (7.2.1) as the inverse Mellin transform of

$$\mu \longmapsto \frac{\gamma(\frac{1}{2}+s, A \otimes \mu)\gamma(1-s, B^{-1} \otimes \mu^{-1})}{\sqrt{\gamma(\frac{1}{2}, S^*)}},$$

evaluated at 1, where $S^*$ is modified from $S^-$ by including the eigenspaces of eigenvalues $\chi_{12}\chi_{13}\chi_{14}^{-1}$, $\chi_{21}\chi_{23}^{-1}\chi_{24}$, $\chi_{41}^{-1}\chi_{42}\chi_{43}$, $\chi_{31}\chi_{32}^{-1}\chi_{34}$ (cf. § 4.4) instead of their inverses; these arise from the $\gamma(\alpha_{2+}), \dots, \gamma(\alpha_{7+})$ factors. To conclude we note that $A \otimes B^{-1}$ is exactly equal to $S^*$ (the underlined ones below are those in $S^*$ but not $S^-$):

$$\begin{array}{c|cccc} & \alpha_{1234567} & \alpha_{34567} & \alpha_{467} & \alpha_7 \\ \hline \alpha_{134567}^{-1} & \underline{\chi_{12}\chi_{13}\chi_{14}^{-1}} & \chi_{12}^{-1}\chi_{13}\chi_{14}^{-1} & \chi_{31}^{-1}\chi_{32}^{-1}\chi_{34}^{-1} & \chi_{31}^{-1}\chi_{32}^{-1}\chi_{34} \\ \alpha_{4567}^{-1} & \chi_{21}^{-1}\chi_{23}^{-1}\chi_{24} & \underline{\chi_{21}\chi_{23}^{-1}\chi_{24}} & \chi_{41}^{-1}\chi_{42}^{-1}\chi_{43} & \chi_{41}^{-1}\chi_{42}^{-1}\chi_{43}^{-1} \\ \alpha_{67}^{-1} & \chi_{21}^{-1}\chi_{23}^{-1}\chi_{24}^{-1} & \chi_{21}\chi_{23}^{-1}\chi_{24}^{-1} & \underline{\chi_{41}^{-1}\chi_{42}\chi_{43}} & \chi_{41}^{-1}\chi_{42}\chi_{43}^{-1} \\ 1 & \chi_{12}\chi_{13}^{-1}\chi_{14}^{-1} & \chi_{12}^{-1}\chi_{13}^{-1}\chi_{14}^{-1} & \chi_{31}\chi_{32}^{-1}\chi_{34}^{-1} & \underline{\chi_{31}\chi_{32}^{-1}\chi_{34}} \end{array} \tag{14.5.2}$$

**14.6. The tetrahedral symbol for the case $F = \mathbb{R}$.** The $F = \mathbb{R}$ case is the most interesting when combined with our discussion in § 14.4. We assume for simplicity that $\chi_{ij}$ ($ij \in \mathbf{O}$) is of the form $|\cdot|^{J_{ij}}$ for $J_{ij} \in i\mathbb{R}$ such that $J_{ij} + J_{ji} = 0$. Using (4.3.1), we also write $J_1 = J_{12}$, $J_2 = J_{31}$, and so on. Then with $a_i, b_i$ ($i = 1, \dots, 4$) as in § 14.4, we have

$$\begin{aligned} &a_1 = J_{12\bar{3}} + \frac{1}{2}, \quad &&a_2 = J_{\bar{1}2\bar{3}} + \frac{1}{2}, \quad &&a_3 = J_{2\bar{4}\bar{6}} + \frac{1}{2}, \quad &&a_4 = J_{246} + \frac{1}{2}, \\ &b_1 = J_{22}, &&b_2 = J_{2\bar{3}\bar{5}6}, &&b_3 = J_{2\bar{3}56}, &&b_4 = 0, \end{aligned}$$

where $J_{12\bar{3}}$ means $J_1 - J_{\bar{2}} - J_3$, and so on. Plugging in the formula (14.4.6) and simplifying using the properties of $\gamma$ and L-factors from § 2, we have

$$\begin{aligned} \{\Pi\}\sqrt{L\Big(\frac{1}{2}, S\Big)} = &-\frac{L(\frac{1}{2}, S_1^*)}{\gamma(J_{\bar{2}\bar{3}\bar{5}6})\gamma(J_{\bar{2}\bar{3}56})\gamma(J_{\bar{2}\bar{2}})} {}_4F_3\left( \begin{matrix} J_{\bar{1}23} + \frac{1}{2}, J_{123} + \frac{1}{2}, J_{24\bar{6}} + \frac{1}{2}, J_{2\bar{4}\bar{6}} + \frac{1}{2} \\ J_{235\bar{6}} + 1, J_{23\bar{5}\bar{6}} + 1, J_{22} + 1 \end{matrix} \,\middle|\, 1 \right) \\ &-\frac{L(\frac{1}{2}, S_2^*)}{\gamma(J_{235\bar{6}})\gamma(J_{55})\gamma(J_{\bar{2}35\bar{6}})} {}_4F_3\left( \begin{matrix} J_{\bar{1}\bar{5}6} + \frac{1}{2}, J_{1\bar{5}6} + \frac{1}{2}, J_{\bar{3}4\bar{5}} + \frac{1}{2}, J_{\bar{3}\bar{4}\bar{5}} + \frac{1}{2} \\ J_{\bar{2}\bar{3}\bar{5}6} + 1, J_{\bar{5}\bar{5}} + 1, J_{2\bar{3}\bar{5}6} + 1 \end{matrix} \,\middle|\, 1 \right) \\ &-\frac{L(\frac{1}{2}, S_3^*)}{\gamma(J_{23\bar{5}\bar{6}})\gamma(J_{\bar{5}\bar{5}})\gamma(J_{\bar{2}3\bar{5}\bar{6}})} {}_4F_3\left( \begin{matrix} J_{\bar{1}56} + \frac{1}{2}, J_{156} + \frac{1}{2}, J_{\bar{3}45} + \frac{1}{2}, J_{\bar{3}\bar{4}5} + \frac{1}{2} \\ J_{\bar{2}\bar{3}56} + 1, J_{55} + 1, J_{2\bar{3}56} + 1 \end{matrix} \,\middle|\, 1 \right) \\ &-\frac{L(\frac{1}{2}, S_4^*)}{\gamma(J_{22})\gamma(J_{2\bar{3}\bar{5}6})\gamma(J_{2\bar{3}56})} {}_4F_3\left( \begin{matrix} J_{\bar{1}\bar{2}3} + \frac{1}{2}, J_{1\bar{2}3} + \frac{1}{2}, J_{\bar{2}4\bar{6}} + \frac{1}{2}, J_{\bar{2}\bar{4}\bar{6}} + \frac{1}{2} \\ J_{\bar{2}\bar{2}} + 1, J_{\bar{2}35\bar{6}} + 1, J_{\bar{2}3\bar{5}\bar{6}} + 1 \end{matrix} \,\middle|\, 1 \right), \end{aligned}$$

where $S_i^*$ is obtained from (14.5.2) by inverting the $i$-th *row*. Note also that the first row of parameters in each ${}_4F_3$ corresponds precisely to the respective row in (14.5.2) up to a universal inversion and half twist.

Similarly, plugging in (14.4.7), we also have

$$\begin{aligned}
\{\Pi\}\sqrt{L\left(\frac{1}{2}, S\right)} = &+\frac{L(\frac{1}{2}, S^*_{1'})}{\gamma(J_{11})\gamma(J_{1\bar{3}46})\gamma(J_{1\bar{3}\bar{4}6})}\,{}_4F_3\left(\begin{matrix} J_{\bar{1}23}+\frac{1}{2}, J_{\bar{1}\bar{5}6}+\frac{1}{2}, J_{\bar{1}56}+\frac{1}{2}, J_{\bar{1}\bar{2}3}+\frac{1}{2} \\ J_{\bar{1}\bar{1}}+1, J_{\bar{1}3\bar{4}\bar{6}}+1, J_{\bar{1}34\bar{6}}+1 \end{matrix}\,\middle|\, 1\right) \\
&+\frac{L(\frac{1}{2}, S^*_{2'})}{\gamma(J_{\bar{1}\bar{1}}\gamma(J_{\bar{1}\bar{3}46})\gamma(J_{\bar{1}\bar{3}\bar{4}6})}\,{}_4F_3\left(\begin{matrix} J_{123}+\frac{1}{2}, J_{1\bar{5}6}+\frac{1}{2}, J_{156}+\frac{1}{2}, J_{1\bar{2}3}+\frac{1}{2} \\ J_{11}+1, J_{13\bar{4}\bar{6}}+1, J_{134\bar{6}}+1 \end{matrix}\,\middle|\, 1\right) \\
&+\frac{L(\frac{1}{2}, S^*_{3'})}{\gamma(J_{\bar{1}3\bar{4}6})\gamma(J_{13\bar{4}6})\gamma(J_{\bar{4}\bar{4}})}\,{}_4F_3\left(\begin{matrix} J_{24\bar{6}}+\frac{1}{2}, J_{\bar{3}4\bar{5}}+\frac{1}{2}, J_{\bar{3}45}+\frac{1}{2}, J_{\bar{2}4\bar{6}}+\frac{1}{2} \\ J_{1\bar{3}4\bar{6}}+1, J_{\bar{1}\bar{3}4\bar{6}}+1, J_{44}+1 \end{matrix}\,\middle|\, 1\right) \\
&+\frac{L(\frac{1}{2}, S^*_{4'})}{\gamma(J_{\bar{1}346})\gamma(J_{1346})\gamma(J_{44})}\,{}_4F_3\left(\begin{matrix} J_{2\bar{4}\bar{6}}+\frac{1}{2}, J_{\bar{3}\bar{4}\bar{5}}+\frac{1}{2}, J_{\bar{3}\bar{4}5}+\frac{1}{2}, J_{\bar{2}\bar{4}\bar{6}}+\frac{1}{2} \\ J_{1\bar{3}\bar{4}\bar{6}}+1, J_{\bar{1}\bar{3}\bar{4}\bar{6}}+1, J_{\bar{4}\bar{4}}+1 \end{matrix}\,\middle|\, 1\right),
\end{aligned}$$

where $S_{i'}^*$ is obtained from (14.5.2) by inverting the $i$-th *column*, and the first row of parameters in each ${}_4F_3$ corresponds to the respective column in (14.5.2) up to a universal inversion and half twist.

## 15. Some proof sketches for § 9

In this section, we provide sketches of proofs of some statements in § 9. We are confident that they turn into real proofs once proper technical care is given, but nevertheless we have not done it. Also, we do in fact wonder if any human being will read this.

**15.1. A sketch of the proof of Proposition 9.1.1.** The following proof is rigorous — but can also be written more easily, as in the traditional theory of the tetrahedral symbol — when $\mathcal{R}$ compact. However, we have chosen to write the proof in a way should be valid in the general case. We say "should be" because there are details we have not tried to fill in: we expect them to be routine but rather tedious and notationally cumbersome to handle. Let us pre-emptively confess to these sins:

- We shall not distinguish between the space of smooth vectors in an $\mathcal{R}$-representation and its Hilbert completion; similarly we will not clearly distinguish between maps that are defined on the smooth part and on the Hilbert completion.
- We will be freely using the theory of unitary decomposition. In particular, this theory handles various set- and measure-theoretic issues that we will not even allude to.

  As a typical example, let $f(\pi)$ be a rule that assigns to each class $[\pi]$ in the unitary dual of $\mathcal{R}$ an element of "the" corresponding unitary representation $\pi$. In this situation is a reasonable way to talk about such $f$s, and there is a reasonable way to talk about them being measurable, and taking the inner products of two such; but all this we will omit.
- Many of the functions below are defined in the sense of measure theory, i.e., off zero measure sets; again, we will ignore this entirely in our language.

- We shall not justify absolute convergence of expressions below. This is the only sin that we think is not venial, because it requires estimates that we did not carry out.

Suppose $\pi, \sigma$ are tempered representations, upon which we fix self-duality pairings. For any tempered representation $\tau \in \mathcal{P}_0$, on which we we also fix a self-duality pairing, we may normalize an invariant functional $\Lambda$ on $\pi \otimes \sigma \otimes \tau$ according to (3.4.3). We dualize it to obtain a map $\pi \otimes \sigma \to \tau$. (The first sin!)

Recall (Lemma 3.4.2) that there are real structures $\pi_{\mathbb{R}}, \sigma_{\mathbb{R}}, \tau_{\mathbb{R}}$ on which the duality structures are inner products. We then define an inner product on $\pi, \sigma, \tau$ by complex-linear extension.

**Lemma 15.1.1.** *The induced map*

$$\pi \otimes \sigma \longrightarrow \int \tau d\tau$$

*extends to an isometry of Hilbert spaces upon completing the left-hand side. On the right, the measure is Plancherel measure; and we integrate over the subset of $\mathcal{P}_0$ with the property that there is a nontrivial invariant functional on $\pi \otimes \sigma \otimes \tau$.*

*Proof.* We prove the corresponding statement for real Hilbert spaces, from which the claim follows by complex-linear extension. This amounts to the following assertion for $v, v' \in \pi_{\mathbb{R}}$ and $w, w' \in \sigma_{\mathbb{R}}$:

$$(v, v')(w, w') = \int d\tau \int_{h \in \mathcal{R}} \sum_{e \in \mathcal{B}_\tau} \langle hv, v' \rangle \langle hw, w' \rangle \langle he, e \rangle dh.$$

where $\mathcal{B}_\tau$ is an orthonormal basis for $\tau_{\mathbb{R}}$, which is a consequence of the Plancherel formula (9.1.1) applied to the function $h \mapsto (hv, v')(hw, w')$. ∎

We now put ourselves in the situation of § 9.1; having fixed self-duality pairings on all the $\pi_{ij}$ we fix inner products as above. *In what follows, $\pi_{\mathbf{12}}, \pi_{\mathbf{23}}, \pi_{\mathbf{34}}$ should be regarded as fixed,* but the remaining $\pi$s will be "varying" — that is, we will not explicitly include in the notation dependence on $\pi_{\mathbf{12}}, \pi_{\mathbf{23}}, \pi_{\mathbf{34}}$.

Applying the lemma (to various choices of $\pi, \sigma$), we find isometries:

$$\pi_{\mathbf{12}} \otimes \pi_{\mathbf{23}} \simeq \int_{\pi_{\mathbf{24}}} \pi_{\mathbf{24}} \implies \pi_{\mathbf{12}} \otimes \pi_{\mathbf{23}} \otimes \pi_{\mathbf{34}} \simeq \int_{\pi_{\mathbf{24}}} \pi_{\mathbf{24}} \otimes \pi_{\mathbf{34}} \simeq \int_{\pi_{\mathbf{14}}, \pi_{\mathbf{24}}} \pi_{\mathbf{14}}. \tag{15.1.1}$$

Here, $\simeq$ means an isometry of Hilbert spaces. The range of integration on the right hand side consists of those $\pi_{\mathbf{14}}, \pi_{\mathbf{24}}$ for which the representation both $\pi_{\mathbf{21}} \otimes \pi_{\mathbf{23}}$ and $\pi_{\mathbf{14}} \otimes \pi_{\mathbf{34}}$ admit nonzero invariant maps to $\pi_{\mathbf{24}}$. For a given $\pi_{\mathbf{14}}$ let us call this set of $\pi_{\mathbf{24}}$ by the name $A(\pi_{\mathbf{14}})$ (it depends on the other $\pi$s too, but we are regarding them as fixed).

We get a similar decomposition with **24** replaced by **13**:

$$\pi_{\mathbf{23}} \otimes \pi_{\mathbf{34}} \simeq \int_{\pi_{\mathbf{13}}} \pi_{\mathbf{13}} \implies \pi_{\mathbf{12}} \otimes \pi_{\mathbf{23}} \otimes \pi_{\mathbf{34}} \simeq \int_{\pi_{\mathbf{12}}} \pi_{\mathbf{12}} \otimes \pi_{\mathbf{13}} \simeq \int_{\pi_{\mathbf{14}}, \pi_{\mathbf{13}}} \pi_{\mathbf{14}}. \tag{15.1.2}$$

where the range of integration consists of pairs $\pi_{\mathbf{14}}, \pi_{\mathbf{13}}$ with the property that both $\pi_{\mathbf{23}} \otimes \pi_{\mathbf{34}}$ and $\pi_{\mathbf{14}} \otimes \pi_{\mathbf{12}}$ admit maps to $\pi_{\mathbf{13}}$. For a given $\pi_{\mathbf{14}}$ let us call this set of $\pi_{\mathbf{13}}$ by the name $B(\pi_{\mathbf{14}})$ (it depends on the other $\pi$s too, but we are regarding them as fixed).

Therefore, there is an isometry

$$\int_{\pi_{14},\pi_{24}} \pi_{14} \simeq \int_{\pi_{14},\pi_{13}} \pi_{14}, \tag{15.1.3}$$

where the ranges of integration are as specified above. Being equivariant for the group action this map necessarily has a rather special form: for each $\pi_{14}$ we must have an isometry

$$L^2(A(\pi_{14})) \simeq L^2(B(\pi_{14})),$$

which is necessarily given by a scalar-valued kernel function $K^{\pi_{14}}(\pi_{24}, \pi_{13})$ on the product $A(\pi_{14}) \times B(\pi_{14})$; explicitly, the isometry of (15.1.3) necessarily has the form

$$f(\pi_{24}, \pi_{14}) \longmapsto \int_{\pi_{24}} K^{\pi_{14}}(\pi_{24}, \pi_{13}) f(\pi_{24}, \pi_{14}), \tag{15.1.4}$$

where the function $f$ is vector-valued: it takes inputs $\pi_{14}$ and $\pi_{24}$ and returns a vector in the space of $\pi_{14}$; on the right, the input parameters are $\pi_{14}$ and $\pi_{13}$ with the output a vector in $\pi_{14}$.

In what follows, we write for typical vectors

$$x \in \pi_{12}, y \in \pi_{23}, z \in \pi_{34}, w \in \pi_{41}.$$

We will denote the image of $x\otimes y\otimes z$ on the right hand side of (15.1.1) by $E^{\pi_{14}}_{\pi_{24}}(x\otimes y\otimes z)$; it is an element of $\pi_{14}$ that depends also on the choice of $\pi_{24}$. Similarly we denote its image on the right hand side of (15.1.2) by $E^{\pi_{14}}_{\pi_{13}}(x \otimes y \otimes z)$. Thus the Es are linear maps

$$E^{\pi_{14}}_{\pi_{24}}, E^{\pi_{14}}_{\pi_{13}} \colon \pi_{12} \otimes \pi_{23} \otimes \pi_{34} \longrightarrow \pi_{14}$$

Let $x, y, z$ vary through orthonormal bases of K-finite vectors for the respective representations. Then $x \otimes y \otimes z$ varies through an orthonormal basis for $\pi_{12} \otimes \pi_{23} \otimes \pi_{34}$. Therefore, $E^{\pi_{14}}_{\pi_{13}}(x \otimes y \otimes z)$ and $E^{\pi_{14}}_{\pi_{24}}(x \otimes y \otimes z)$, considered as $\pi_{14}$-valued functions of $(\pi_{14}, \pi_{13})$ or $(\pi_{14}, \pi_{24})$, form orthonormal bases for the two Hilbert spaces appearing on corresponding sides of (15.1.3). The unitary transformation sending one basis to the other sends a function $f(\pi_{24}, \pi_{14})$, taking values in $\pi_{14}$, to

$$f' \colon (\pi_{13}, \pi_{14}) \longmapsto \sum_{x,y,z} E^{\pi_{14}}_{\pi_{13}}(x \otimes y \otimes z) \int_{\pi_{24},\pi'_{14}} \langle f(\pi_{24}, \pi'_{14}), E^{\pi'_{14}}_{\pi_{24}}(x \otimes y \otimes z)\rangle.$$

Let $f(\pi_{24}, \pi_{14}), f'(\pi_{13}, \pi_{14})$ be arbitrary, but "real-valued," i.e. valued in a fixed real form $\pi_{14,\mathbb{R}}$; this permits us to ignore complex conjugates in inner products. Compare the above equation with (15.1.4) to conclude

$$\begin{aligned}\int_{\pi_{14},\pi_{24}} & K^{\pi_{14}}(\pi_{24}, \pi_{13})\langle f(\pi_{24}, \pi_{14}), f'(\pi_{13}, \pi_{14})\rangle \\ &= \int_{\pi_{24},\pi_{14},\pi'_{14}} \sum_{x,y,z} \langle f(\pi_{24}, \pi'_{14}) \otimes f'(\pi_{13}, \pi_{14}), E^{\pi'_{14}}_{\pi_{24}}(x \otimes y \otimes z) \otimes E^{\pi_{14}}_{\pi_{13}}(x \otimes y \otimes z)\rangle.\end{aligned}$$

Since $f$ is arbitrary, the same equality still holds without integrating over $\pi_{24}$; so

$$\int_{\pi_{14}} K^{\pi_{14}}(\pi_{24}, \pi_{13}) \langle f(\pi_{24}, \pi_{14}), f'(\pi_{13}, \pi_{14}) \rangle$$
$$= \int_{\pi_{14}, \pi'_{14}} \sum_{x,y,z} \langle f(\pi_{24}, \pi'_{14}) \otimes f'(\pi_{13}, \pi_{14}), E^{\pi'_{14}}_{\pi_{24}}(x \otimes y \otimes z) \otimes E^{\pi_{14}}_{\pi_{13}}(x \otimes y \otimes z) \rangle. \quad (15.1.5)$$

Previously we fixed $\pi_{12}, \pi_{23}, \pi_{34}$. We now additionally fix $\pi_{13}$ and $\pi_{24}$, so that we are now regarding *all the* $\pi_{ij}$ *for* $\{ij\} \neq \{\mathbf{14}\}$ *as fixed*, and will not include in the notation dependence on these representations. Having fixed these, we put

$$K^{\pi_{14}} := K^{\pi_{14}}(\pi_{24}, \pi_{13}).$$

Again, here, we have a measure-theoretic issue, since $K$ is defined only off a set of measure zero; in actuality, all the statements hold almost everywhere, and are to be extended by continuity.

Abridge $\pi_{14}$ to $\pi$ and $\pi'_{14}$ to $\pi'$, switch the letters $f, f'$ for nicer typography, and rewrite (15.1.5) as:

$$K^{\pi} \cdot \int_{\pi} \langle f(\pi), f'(\pi) \rangle = \int_{\pi, \pi'} \sum_{x,y,z} \langle f'(\pi') \otimes f(\pi), E^{\pi'}_{\pi_{24}}(x \otimes y \otimes z) \otimes E^{\pi}_{\pi_{13}}(x \otimes y \otimes z) \rangle. \quad (15.1.6)$$

Let us reinterpret the right hand side as follows. Consider the representation

$$\Pi^*_G = (\pi' \otimes \pi) \otimes \bigotimes_{ij \in \mathbf{E} - \{\mathbf{14}\}} \pi_{ij} \otimes \pi_{ij}$$

i.e., similar to $\Pi_G$ but now taking $\pi_{14} = \pi, \pi_{41} = \pi'$, that is, two previously isomorphic copies of $\pi_{14}$ are taken to be distinct representations. We can define as before $\Lambda^{H*}$ in the dual of $\Pi^*_G$ by tensoring the invariant trilinear functionals on triples of representations indexed by edges sharing a common vertex. Note that both $\Pi^*_G$ and $\Lambda^{H*}$ depend on both $\pi$ and $\pi'$, but to simplify the notation we will not explicitly denote this.

Let $\Pi^*_{G,\mathrm{big}}$ be the same expression integrated over $\pi, \pi'$, and define $\tilde{\Lambda}^H$ in the dual of $\Pi^*_{G,\mathrm{big}}$ by similarly integrating $\Lambda^{H*}$, where, in both cases, all integrals are taken with respect to Plancherel measure in both $\pi$ and $\pi'$:

$$\Pi^*_{G,\mathrm{big}} = \int_{\pi,\pi'} \Pi^*_G, \quad \tilde{\Lambda}^H = \int_{\pi,\pi'} \Lambda^{H*}.$$

Consider the expression $\sum_x x \otimes x \in \pi_{12} \otimes \pi_{12}$ and its analogues for $\pi_{23}$ and so on; let $\Delta$ be obtained by tensoring together all these expressions for all $ij \in \mathbf{E} - \{\mathbf{14}\}$. Finally, let $Y := \int f' \otimes \int f \in \int_{\pi,\pi'} \pi' \otimes \pi$. Then we may rewrite (15.1.6) as a pairing inside $\Pi^*_{G,\mathrm{big}}$:

$$\langle Y \otimes \Delta, \tilde{\Lambda}^H \rangle_{\Pi^*_{G,\mathrm{big}}}. \quad (15.1.7)$$

To proceed we must observe an alternative way of writing the tetrahedral symbol.

**Lemma 15.1.2.** *Notation as above, we have*

$$(Y \otimes \Delta, \tilde{\Lambda}^H) = \int_{\pi} \langle f(\pi), f'(\pi) \rangle \{\Pi\} \quad (15.1.8)$$

*where* $\{\Pi\}$ *takes as arguments the fixed* $\pi_{ij}$ *for* $ij \neq \mathbf{14}$, *and* $\pi_{14} = \pi$.

Given this, it is easy to finish the proof: Starting at (15.1.6) we find

$$\int_\pi K^\pi \langle f(\pi), f'(\pi)\rangle = (15.1.7) = \int_\pi \{\Pi\}\langle f(\pi), f'(\pi)\rangle,$$

and, this being valid for all choices of $f, f'$, we see that $\{\Pi\}$ coincides with $K^\pi$, which is the kernel of an isometric isomorphism of $L^2(A(\pi))$ and $L^2(B(\pi))$, as desired.

*Proof of Lemma 15.1.2.* First let us prove an easier-to-grasp version. Suppose that we can find a functional $\lambda^D$ on $\Pi_G$ such that

$$\int_{\delta \in D\cap H} \delta \cdot \lambda^D = \Lambda^D, \tag{15.1.9}$$

where we can think of $\lambda^D$ as an un-averaging of $\Lambda^D$. Then we have, simply,

$$\{\Pi\} = (\Lambda^H, \lambda^D), \tag{15.1.10}$$

where the left hand side means $\lim_i \Lambda^H(\nu_i)$ when $\lambda^D = \lim_i(\nu_i, -)$. To prove (15.1.10) we choose $\nu_H \in \Pi_G$ with the property that $\Lambda^H(\nu_H) = 1$ and successively rewrite $\{\Pi\}$ via

$$\begin{aligned}(\Lambda^H)'(\nu_H) &= \int_{h\in D\cap H\backslash H} \Lambda^D(h\nu_H) = \int_{h\in H} \lambda^D(h\nu_H)\\ &= \int_H \lim_i(\nu_i, h\nu_H) = \lim_i \int_H (\nu_i, h\nu_H) = \lim_i \Lambda^H(\nu_i).\end{aligned}$$

Write $u(\pi) = \langle f(\pi), f'(\pi)\rangle$, so that the desired (15.1.8) can be written as $\int_\pi u(\pi)\{\Pi\} = (\tilde{\Lambda}^H, Y\otimes\Delta)$. Note that this is a version of (15.1.10) with $\Pi_G$ replaced by $\Pi^*_{G,\mathrm{big}}$ and with $Y\otimes\Delta$ playing the role of $\lambda^D$, and we use similar reasoning to prove it. Note, first of all, the following analogue of (15.1.9):

$$\int_{\delta\in D\cap H} \delta\cdot(Y\otimes\Delta) = \int_\pi u(\pi)\Lambda^D. \tag{15.1.11}$$

This is a consequence of the Plancherel formula, which gives the analogue of the "Schur orthogonality relations" in the current context. The point is that the averaged vector $\int_{h\in\mathcal{R}} h\cdot Y$ represents, on $\int_{\pi,\pi'} \pi\otimes\pi'$, the functional that corresponds to restricting to the diagonal, contracting, and integrating against $\langle f, f'\rangle$ times the Plancherel measure. Now, to prove (15.1.8) we "do the same thing in a family," proceeding as before but replacing $\nu_H$ now by a $(\pi,\pi')$-dependent family of vectors $\nu_H(\pi,\pi') \in \Pi^*_G$, chosen to have the property that $\Lambda^H(\nu_H) = 1$ for all $\pi, \pi'$; write $\mathbf{v}_H$ for the integrated vector $\int \nu_H(\pi,\pi') \in \Pi^*_{G,\mathrm{big}}$. The result is

$$\int_{h\in H} (Y\otimes\Delta, h\mathbf{v}_H)_{\Pi^*_{G,\mathrm{big}}} \overset{(15.1.11)}{=} \int_{D\cap H\backslash H}\int_\pi u(\pi)(\Lambda^D, h\nu_H(\pi,\pi)) = \int_\pi u(\pi)\{\Pi\}.$$

This finishes the proof. ∎

**15.2. Relationship between geometric representation theory and Theorem 5.2.1, a sketch.** We follow notation as in § 9.7, and will write $\mathfrak{f} = \mathbb{F}_q((t)) \supset \mathfrak{o} = \mathbb{F}_q[[t]]$ for the analogues of $F, \mathcal{O}$ over $\mathbb{F}_q$. We write for this argument

$$\alpha = (1 - q^{-2}) = L(2)^{-1}.$$

Let us consider, as in (8.1.1), integration over $(H \cap D)\backslash H$ as defining an averaging intertwiner Av from $X = D\backslash G$ to $Y = H\backslash G$. We will use the normalized version

$$I_{\mathrm{aut}} = \alpha^3 \mathrm{Av},$$

which comes from the fact that it is geometrically natural to use the "point-counting" measure where $\mathrm{PGL}_2(\mathfrak{o})$ has mass $1 - 1/q^2$, coming from the point count over $\mathbb{F}_q$. We regard $I_{\mathrm{aut}}$ an interwiner from functions on $Y_{\mathfrak{f}}/G_{\mathfrak{o}}$ to $X_{\mathfrak{f}}/G_{\mathfrak{o}}$. We will give these spaces $X_{\mathfrak{f}}/G_{\mathfrak{o}}$ and $Y_{\mathfrak{f}}/G_{\mathfrak{o}}$ also the point-counting measures; for example the measure of $X_{\mathfrak{o}}/G_{\mathfrak{o}}$ is then equal to $\alpha^{-6}$, whereas the measure of $X_{\mathfrak{o}}/G_{\mathfrak{o}}$ is equal to $\alpha^{-4}$.

Let $\check{D}_0$ be a maximal compact subgroup of the dual group of $\check{D}$, which we regard as embedded in $\check{G}$ in the diagonal fashion. For any $\sigma \in \check{D}_0$ let $\varphi_\sigma^Y$ be the corresponding normalized spherical function on $Y_{\mathfrak{f}}/G_{\mathfrak{o}}$, whose value at the identity equals 1; and let $\varphi_\sigma^X$ be the normalized spherical function on $X_{\mathfrak{f}}/G_{\mathfrak{o}}$; its value at the identity coset equals, instead, $\alpha^{-8}\sqrt{\frac{L(\frac{1}{2},S)}{L(1,\check{\mathfrak{g}})}}$, see (6.2.1). By definition,

$$I_{\mathrm{aut}} \varphi_\sigma^Y = \alpha^3 \{\Pi\} \varphi_\sigma^X,$$

where $\Pi = \Pi(\sigma)$ is the unramified representation with parameter $\sigma$. By the Plancherel formula, we get

$$1_{Y_{\mathfrak{o}}} = \int_{\sigma \in \check{D}_0} \langle 1_{Y_{\mathfrak{o}}}, \varphi_\sigma^Y \rangle_{Y_{\mathfrak{f}}/G_{\mathfrak{o}}} \varphi_\sigma^Y \cdot \alpha^{12} L(1, \check{\mathfrak{g}})^{\frac{1}{2}} \tag{15.2.1}$$

and applying $I_{\mathrm{aut}}$ and pairing we find

$$\begin{aligned} \langle 1_{X_{\mathfrak{o}}}, I_{\mathrm{aut}} 1_{Y_{\mathfrak{o}}} \rangle &= \int_{\check{D}_0} \alpha^3 \{\Pi\} \underbrace{\langle 1_{Y_{\mathfrak{o}}}, \varphi_\sigma^Y \rangle_{Y_{\mathfrak{f}}/G_{\mathfrak{o}}} \langle 1_{X_{\mathfrak{o}}}, \phi_\sigma^X \rangle_{X_{\mathfrak{f}}/G_{\mathfrak{o}}}}_{\alpha^{-10}\varphi_\sigma^X(1)} \alpha^{12} \sqrt{L(1, \check{\mathfrak{g}})} \\ &= \alpha^{-3} \int_{\check{D}_0} \{\Pi\} \sqrt{L\Big(\frac{1}{2}, S\Big)}. \end{aligned}$$

where the exponent $-3$ arises from $3 - 10 - 8 + 12$. There is a similar formula with a Hecke operator inserted.

Let us compare this with what we get from the conjecture enunciated around (9.7.1). Computing Hom-spaces [19] and then passing to Frobenius trace, we find

$$\langle 1_{X_o}, I_{\mathrm{aut}} 1_{Y_o}\rangle = \mathrm{Tr}(q^{-\frac{1}{2}}\sigma, k[P]^{\check{D}}) = \int_{\sigma \in \check{D}_0} \text{character of } q^{-\frac{1}{2}}\sigma \text{ on } k[P]. \tag{15.2.2}$$

Here $\check{D}_0$ is a maximal compact subgroup of $\check{D}$, and the $q^{-\frac{1}{2}}$ arises from interpretation the effect of shearing. Again, there is a corresponding formula with a Hecke operator inserted. Comparing (15.2.1) and (15.2.2), and moreover the versions with Hecke modifications, one can identify the integrands, and not merely the integrals; so we find

$$\{\Pi\} \cdot \sqrt{L\Big(\frac{1}{2}, S\Big)} = \alpha^3 \cdot \text{character of } k[P] \text{ at } q^{-\frac{1}{2}}\sigma$$

which agrees with Theorem 5.2.1.

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[19]Now, if we apply $I_{\mathrm{spec}}$ to the structure sheaf on $\check{M}/\check{G}$, the result is simply the structure sheaf of $L/\check{G}$, but considered as a sheaf on $\check{N}/\check{G}$. Therefore

$$\mathrm{Hom}(\mathcal{O}_{\check{N}}, I_{\mathrm{spec}}\mathcal{O}_{\check{M}}) \simeq k[\check{L}]^{\check{G}} \simeq k[P]^{\check{D}}$$

where we used (8.2.1). Pass this equivalence to the automorphic side. The structure sheaves above correspond to constant sheaves on $X_o$ and $Y_o$, and so the above formula computes $\mathrm{Hom}(1_{X_o}, I_{\mathrm{aut}} 1_{Y_o})$.

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